Notes for Signals and Systems

Version 1.0

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These notes were developed for use in 520.214, *Signals and Systems*, Department of Electrical and Computer Engineering, Johns Hopkins University, over the period 2000 – 2005. As indicated by the *Table of Contents*, the notes cover traditional, introductory concepts in the time domain and frequency domain analysis of signals and systems. Not as complete or polished as a book, though perhaps subject to further development, these notes are offered on an *as is* or *use at your own risk* basis.

Prerequisites for the material are the arithmetic of complex numbers, differential and integral calculus, and a course in electrical circuits. (Circuits are used as examples in the material, and the last section treats circuits by Laplace transform.) Concurrent study of multivariable calculus is helpful, for on occasion a double integral or partial derivative appears. A course in differential equations is not required, though some very simple differential equations appear in the material.

The material includes links to demonstrations of various concepts. These and other demonstrations can be found at http://www.jhu.edu/~signals/ .

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Notes for Signals and Systems

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0.1 Introductory Comments

What is "Signals and Systems?" Easy, but perhaps unhelpful answers, include

- the α and the ω,
- the question and the answer,
- the fever and the cure,
- calculus and complex arithmetic for fun and profit,

More seriously, signals are functions of time (continuous-time signals) or sequences in time (discrete-time signals) that presumably represent quantities of interest. Systems are operators that accept a given signal (the *input signal*) and produce a new signal (the *output signal*). Of course, this is an abstraction of the processing of a signal.

From a more general viewpoint, systems are simply functions that have domain and range that are sets of functions of time (or sequences in time). It is traditional to use a fancier term such as *operator* or *mapping* in place of *function*, to describe such a situation. However we will not be so formal with our viewpoints or terminologies. Simply remember that signals are abstractions of time-varying quantities of interest, and systems are abstractions of processes that modify these quantities to produce new time-varying quantities of interest.

These notes are about the mathematical representation of signals and systems. The most important representations we introduce involve the *frequency domain* – a different way of looking at signals and systems, and a complement to the time-domain viewpoint. Indeed engineers and scientists often think of signals in terms of frequency content, and systems in terms of their effect on the frequency content of the input signal. Some of the associated mathematical concepts and manipulations involved are challenging, but the mathematics leads to a new way of looking at the world!

0.2 Background in Complex Arithmetic

We assume easy familiarity with the arithmetic of complex numbers. In particular, the *polar form* of a complex number c, written as

$$c = |c| e^{j\angle c}$$

is most convenient for multiplication and division, e.g.,

$$c_1 c_2 = |c_1| e^{j\angle c_1} |c_2| e^{j\angle c_2} = |c_1||c_2| e^{j(\angle c_1 + \angle c_2)}$$

The *rectangular form* for c, written

$$c = a + jb$$

where a and b are real numbers, is most convenient for addition and subtraction, e.g.,

$$c_1 + c_2 = a_1 + jb_1 + a_2 + jb_2 = (a_1 + a_2) + j(b_1 + b_2)$$

Of course, connections between the two forms of a complex number c include

$$|c| = |a + jb| = \sqrt{a^2 + b^2}\ , \quad \angle c = \angle(a + jb) = \tan^{-1}(b/a)$$

and, the other way round,

$$a = \text{Re}\{c\} = |c|\cos(\angle c)\,, \quad b = \text{Im}\{c\} = |c|\sin(\angle c)$$

Note especially that the quadrant ambiguity of the inverse tangent must be resolved in making these computations. For example,

$$\angle(1-j) = \tan^{-1}(-1/1) = -\pi/4$$

while

$$\angle(-1+j) = \tan^{-1}(1/(-1)) = 3\pi/4$$

It is important to be able to mentally compute the sine, cosine, and tangent of angles that are integer multiples of $\pi/4$, since many problems will be set up this way to avoid the distraction of calculators.

You should also be familiar with Euler's formula,

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

and the complex exponential representation for trigonometric functions:

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}\,, \quad \sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Notions of complex numbers extend to notions of complex-valued functions (of a real variable) in the obvious way. For example, we can think of a complex-valued function of time, $x(t)$, in the rectangular form

$$x(t) = \text{Re}\{x(t)\} + j\,\text{Im}\{x(t)\}$$

In a simpler notation this can be written as

$$x(t) = x_R(t) + j\,x_I(t)$$

where $x_R(t)$ and $x_I(t)$ are real-valued functions of t.
Or we can consider polar form,

$$x(t) = |x(t)|\,e^{j\angle x(t)}$$

where $|x(t)|$ and $\angle x(t)$ are real-valued functions of t (with, of course, $|x(t)|$ nonnegative for all t). In terms of these forms, multiplication and addition of complex functions can be carried out in the obvious way, with polar form most convenient for multiplication and rectangular form most convenient for addition.

In all cases, signals we encounter are functions of the real variable t. That is, while signals that are complex-valued functions of t, or some other real variable, will arise as mathematical conveniences, we will not deal with functions of a complex variable until near the end of the course.

0.3 Analysis Background

We will use the notation $x[n]$ for a real or complex-valued sequence (discrete-time signal) defined for integer values of *n*. This notation is intended to emphasize the similarity of our treatment of functions of a continuous variable (time) and our treatment of sequences (in time). But use of the square brackets is intended to remind us that the similarity should not be overdone!

Summation notation, for example,

$$\sum_{k=1}^{3} x[k] = x[1] + x[2] + x[3]$$

is extensively used. Of course, addition is commutative, and so we conclude that

$$\sum_{k=1}^{3} x[k] = \sum_{k=3}^{1} x[k]$$

Care must be exercised in consulting other references since some use the convention that a summation is zero if the upper limit is less than the lower limit. And of course this summation limit reversal is not to be confused with the integral limit reversal formula:

$$\int_{1}^{3} x(t)\, dt = -\int_{3}^{1} x(t)\, dt$$

It is important to manage summation indices to avoid collisions. For example,

$$z[k] \sum_{k=1}^{3} x[k]$$

is not the same thing as

$$\sum_{k=1}^{3} z[k]\, x[k]$$

But it is the same thing as

$$\sum_{j=1}^{3} z[k]\, x[j]$$

All these observations are involved in changes of variables of summation. A typical case is

$$\sum_{k=1}^{3} x[n-k]$$

Let $j = n - k$ (relying on context to distinguish the new index from the imaginary unit j) to rewrite the sum as

$$\sum_{j=n-1}^{n-3} x[j] = \sum_{j=n-3}^{n-1} x[j]$$

Sometimes we will encounter multiple summations, often as a result of a product of summations, for example,

$$\left(\sum_{k=1}^{4} x[k]\right)\left(\sum_{j=0}^{5} z[j]\right) = \sum_{k=1}^{4} \sum_{j=0}^{5} x[k]\, z[j] = \sum_{j=0}^{5} \sum_{k=1}^{4} x[k]\, z[j]$$

The order of summations here is immaterial. But, again, look ahead to be sure to avoid index collisions by changing index names when needed. For example, write

$$\left(\sum_{k=1}^{4} x[k]\right)\left(\sum_{k=0}^{5} z[k]\right) = \left(\sum_{k=1}^{4} x[k]\right)\left(\sum_{j=0}^{5} z[j]\right)$$

before proceeding as above.

These considerations also arise, in slightly different form, when integral expressions are manipulated. For example, changing the variable of integration in the expression

$$\int_0^t x(t-\tau)\, d\tau$$

to $\sigma = t - \tau$ gives

$$\int_t^0 x(\sigma)\,(-d\sigma) = \int_0^t x(\sigma)\, d\sigma$$

We encounter multiple integrals on rare occasions, usually as a result of a product of integrals, and collisions of integration variables must be avoided by renaming. For example,

$$\left(\int_0^3 x(t)\, dt\right)\left(\int_{-1}^3 z(t)\, dt\right) = \left(\int_0^3 x(t)\, dt\right)\left(\int_{-1}^3 z(\tau)\, d\tau\right)$$
$$= \int_0^3 \int_{-1}^3 x(t)\, z(\tau)\, dt\, d\tau$$

The Fundamental Theorem of Calculus arises frequently:

$$\frac{d}{dt} \int_{-\infty}^t x(\tau)\, d\tau = x(t)$$

For finite sums, or integrals of well-behaved (e.g. continuous) functions with finite integration limits, there are no particular technical concerns about existence of the sum or integral, or interchange of order of integration or summation. However, for infinite sums or improper integrals (over an infinite range) we should be concerned about convergence and then about various manipulations involving change of order of operations. However, we will be a bit cavalier about this. For summations such as

$$\sum_{k=-\infty}^{\infty} x[k]$$

a rather obvious necessary condition for convergence is that $|\, x[k]\,| \to 0$ as $k \to \pm\infty$. Typically we will not worry about general sufficient conditions, rather we leave consideration of convergence to particular cases.

For integrals such as

$$\int_{-\infty}^{\infty} x(t)\, dt$$

an obvious necessary condition for convergence is that $|\, x(t)\,| \to 0$ as $t \to \pm\infty$, but again further details will be ignored. We especially will ignore conditions under which the order of a double (infinite) summation can be interchanged, or the order of a double (improper) integral can be interchanged. Indeed, many of the mathematical magic tricks that appear in our subject are explainable only by taking a very rigorous view of these issues. Such rigor is beyond our scope.

For complex-valued functions of time, operations such as differentiation and integration are carried out in the usual fashion with j viewed as a constant. It sometimes helps to think of the function in rectangular form to justify this view: for example, if $x(t) = x_R(t) + j\, x_I(t)$, then

$$\int_{-\infty}^{t} x(\tau)\,d\tau = \int_{-\infty}^{t} x_R(\tau)\,d\tau + j\int_{-\infty}^{t} x_I(\tau)\,d\tau$$

Similar comments apply to complex summations and sequences.

Pathologies that sometimes arise in the calculus, such as everywhere continuous but nowhere differentiable functions (signals), are of no interest to us! On the other hand, certain generalized notions of functions, particularly the impulse function, will be very useful for representing special types of signals and systems. Because we do not provide a careful mathematical background for generalized functions, we will take a very formulaic approach to working with them. Impulse functions aside, fussy matters such as signals that have inconvenient values at isolated points will be handled informally by simply adjusting values to achieve convenience.

Example Consider the function

$$x(t) = \begin{cases} 1, & t = 0 \\ 0, & else \end{cases}$$

Certainly the integral of $x(t)$ between any two limits, is zero – there being no area under a single point. The derivative of $x(t)$ is zero for any $t \neq 0$, but the derivative is undefined at $t = 0$, there being no reasonable notion of "slope." How do we deal with this? The answer is to view $x(t)$ as equivalent to the identically-zero function. Indeed, we will happily adjust the value of a function at isolated values of t for purposes of convenience and simplicity.

In a similar fashion, consider

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

which probably is familiar as the *unit-step function*. What value should we assign to $u(0)$? Again, the answer is that we choose $u(0)$ for convenience. For some purposes, setting $u(0) = 1/2$ is most suitable, for other purposes $u(0) = 1$ is best. But in every instance we freely choose the value of $u(0)$ to fit the purpose at hand. The derivative of $u(t)$ is zero for all $t \neq 0$, but is undefined in the usual calculus sense at $t = 0$. However there is an intuitive notion that a jump upward has infinite slope (and a jump downward has slope $-\infty$). We will capture this notion using generalized functions and a notion of generalized calculus in the sequel. By comparison, the signal $x(t)$ in the example above effectively exhibits two simultaneous jumps, and there is little alternative than to simplify $x(t)$ to the zero signal.

Except for generalized functions, to be discussed in the sequel, we typically work in the context of piecewise-continuous functions, and permit only simple, finite jumps as discontinuities.

Exercises

1. Compute the polar form of the complex numbers $e^{j(1+j)}$ and $(1+j)e^{-j\pi/2}$.

2. Compute the rectangular form of the complex numbers $2\,e^{j5\pi/4}$ and $e^{-j\pi} + e^{j6\pi}$.

3. Evaluate, the easy way, the magnitude $|(2-j2)^3|$ and the angle $\angle(-1-j)^2$.

4. Using Euler's relation, $e^{j\theta} = \cos\theta + j\sin\theta$, derive the expression

$$\cos\theta = \tfrac{1}{2}e^{j\theta} + \tfrac{1}{2}e^{-j\theta}$$

5. If z_1 and z_2 are complex numbers, and a star denotes complex conjugate, express the following quantities in terms of the real and imaginary parts of z_1 and z_2 :

$$\mathrm{Re}[z_1 - z_1^*], \quad \mathrm{Im}[z_1 z_2], \quad \mathrm{Re}[z_1 / z_2]$$

6. What is the relationship among the three expressions below?

$$\int_{-\infty}^{\infty} x(\sigma)\, d\sigma, \quad \int_{-\infty}^{\infty} x(-\sigma)\, d\sigma, \quad 2\int_{-\infty}^{\infty} x(2\sigma)\, d\sigma$$

7. Simplify the three expressions below.

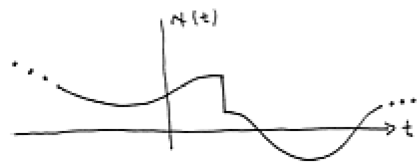
$$\frac{d}{dt}\int_{0}^{t} x(\sigma)\, d\sigma, \quad \frac{d}{dt}\int_{-t}^{0} x(\sigma)\, d\sigma, \quad \frac{d}{d\sigma}\int_{t}^{0} x(\sigma)\, d\sigma$$

Notes for Signals and Systems

1.1 Mathematical Definitions of Signals

A *continuous-time signal* is a quantity of interest that depends on an independent variable, where we usually think of the independent variable as time. Two examples are the voltage at a particular node in an electrical circuit and the room temperature at a particular spot, both as functions of time. A more precise, mathematical definition is the following.

A *continuous-time signal* is a function $x(t)$ of the real variable *t* defined for $-\infty < t < \infty$. A crude representation of such a signal is a sketch, as shown.



On planet earth, physical quantities take on real numerical values, though it turns out that sometimes it is mathematically convenient to consider *complex-valued functions* of t. However, the default is real-valued $x(t)$, and indeed the type of sketch exhibited above is valid only for real-valued signals. A sketch of a complex-valued signal $x(t)$ requires an additional dimension or multiple sketches, for example, a sketch of the real part, $\text{Re}\{x(t)\}$, versus t and a sketch of the imaginary part, $\text{Im}\{x(t)\}$, versus t.

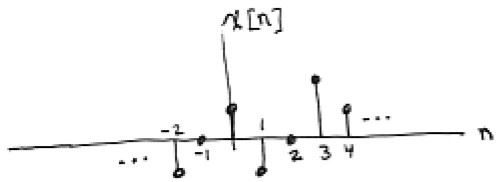
Remarks:

- A continuous-time signal is not necessarily a continuous function, in the sense of calculus. Discontinuities (jumps) in a signal are indicated by a vertical line, as drawn above.
- The default domain of definition is always the whole real line – a convenient abstraction that ignores various big-bang theories. We use ellipses as shown above to indicate that the signal "continues in a similar fashion," with the meaning presumably clear from context. If a signal is of interest only over a particular interval in the real line, then we usually define it to be zero outside of this interval so that the domain of definition remains the whole real line. Other conventions are possible, of course. In some cases a signal defined on a finite interval is extended to the whole real line by endlessly repeating the signal (in both directions).
- The independent variable need not be time, it could be distance, for example. But for simplicity we will always consider it to be time.
- An important subclass of signals is the class of *unilateral* or *right-sided* signals that are zero for negative arguments. These are used to represent situations where there is a definite starting time, usually designated $t = 0$ for convenience.

A *discrete-time signal* is a sequence of values of interest, where the integer index can be thought of as a time index, and the values in the sequence represent some physical quantity of interest. Because many discrete-time signals arise as equally-spaced samples of a continuous-time signal, it is often more convenient to think of the index as the "sample number." Examples are the closing Dow-Jones stock average each day and the room temperature at 6 pm each day. In these cases, the sample number would be day 0, day 1, day 2, and so on.

We use the following mathematical definition.

A *discrete-time signal* is a sequence $x[n]$ defined for all integers $-\infty < n < \infty$. We display $x[n]$ graphically as a string of lollypops of appropriate height.



Of course there is no concept of continuity in this setting. However, all the remarks about domains of definition extend to the discrete-time case in the obvious way. In addition, complex-valued discrete-time signals often are mathematically convenient, though the default assumption is that $x[n]$ is a real sequence.

In due course we discuss converting a signal from one domain to the other – sampling and reconstruction, also called *analog-to-digital* (A/D) and *digital-to-analog* (D/A) conversion.

1.2 Elementary Operations on Signals

Several basic operations by which new signals are formed from given signals are familiar from the algebra and calculus of functions.

- *Amplitude Scale*: $y(t) = a\, x(t)$, where a is a real (or possibly complex) constant

- *Amplitude Shift*: $y(t) = x(t) + b$, where b is a real (or possibly complex) constant

- *Addition*: $y(t) = x(t) + z(t)$

- *Multiplication*: $y(t) = x(t)\, z(t)$

With a change in viewpoint, these operations can be viewed as simple examples of *systems*, a topic discussed at length in the sequel. In particular, if a and b are assumed real, and $z(t)$ is assumed to be a fixed, real signal, then each operation describes a system with input signal $x(t)$ and output signal $y(t)$. This viewpoint often is not particularly useful for such simple situations, however.

The description of these operations for the case of discrete-time signals is completely analogous.

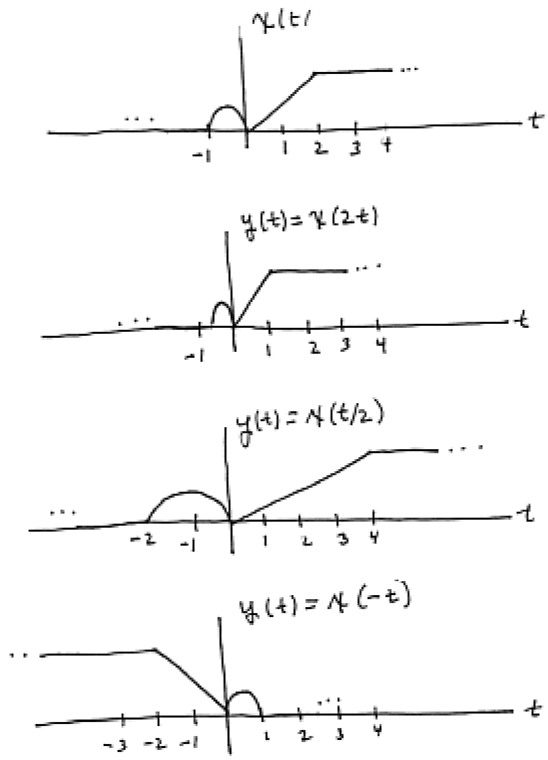
1.3 Elementary Operations on the Independent Variable

Transformations of the independent variable are additional, basic operations by which new signals are formed from a given signal. Because these involve the independent variable, that is, the argument (t), the operations sometimes subtly involve our customary notation for functions.

These operations can be viewed as somewhat less simple examples of systems, and sometimes such an alternate view is adopted.

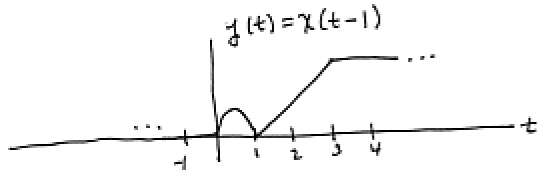
As is typical in calculus, we use the notation $x(t)$ to denote both the entire signal, and the value of the signal at a value of the independent variable called t. The interpretation depends on context. This is simpler than adopting a special notation, such as $x(\bullet)$, to describe the entire signal. Subtleties that arise from our dual view will be discussed in the particular context.

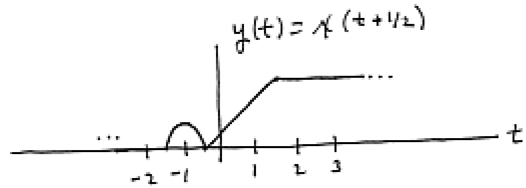
- *Time Scale*: Suppose $y(t) = x(at)$ where a is a real constant. By sketching simple examples, it becomes clear that if $a > 1$, the result is a time-compressed signal, and if $0 < a < 1$, the result is time dilation. Of course, the case $a = 0$ is trivial, giving the constant signal $y(t) = x(0)$ that is only slightly related to $x(t)$. For $a \neq 0$, $x(t)$ can be recovered from $y(t)$. That is, the operation is invertible. If $a < 0$, then there is a time reversal, in addition to compression or dilation. The recommended approach to sketching time-scaled signals is simply to evaluate $y(t)$ for a selection of values of t until the result becomes clear. For example,



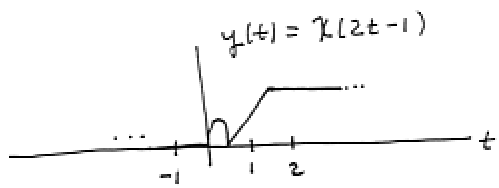
Notice that in addition to compression or dilation, the `beginning time' or `ending time' of a pulse-type signal will be changed in the new time scale.

- *Time Shift*: Suppose $y(t) = x(t - T)$ where T is a real constant. If $T > 0$, the shift is a right shift in time, or a time delay. If T is negative, we have a left shift, or a time advance. For example,

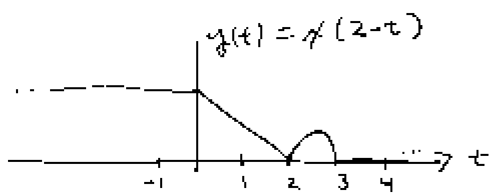




- *Combination Scale and Shift*: Suppose $y(t) = x(at - T)$. It is tempting to think about this as two operations in sequence -- a scale followed by a shift, or a shift followed by a scale. This is dangerous in that a wrong choice leads to incorrect answers. The recommended approach is to ignore shortcuts, and figure out the result by brute-force graphical methods: substitute various values of *t* until $y(t)$ becomes clear. Continuing the example,



Example: The most important scale and shift combination for the sequel is the case where $a = -1$, and the sign of *T* is changed to write $y(t) = x(T - t)$. This is accomplished graphically by reversing time and then shifting the reversed signal *T* units to the right if $T > 0$, or to the left if $T < 0$. We refer to this transformation as the *flip and shift.* For example,



The flip and shift operation can be explored in the applet below. However, you should verify the interpretation of the flip and shift by hand sketches of a few examples.

flip and shift

1.4 Energy and Power Classifications

The *total energy* of a continuous-time signal $x(t)$, where $x(t)$ is defined for $-\infty < t < \infty$, is

$$E_\infty = \int_{-\infty}^{\infty} x^2(t)dt = \lim_{T\to\infty} \int_{-T}^{T} x^2(t)\, dt$$

In many situations, this quantity is proportional to a physical notion of energy, for example, if $x(t)$ is the current through, or voltage across, a resistor. If a signal has finite energy, then the signal values must approach zero as *t* approaches positive and negative infinity.

The *time-average power* of a signal is

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)\, dt$$

For example the constant signal $x(t) = 1$ (for all *t*) has time-average power of unity.

With these definitions, we can place most, but not all, continuous-time signals into one of two classes:

- An *energy signal* is a signal with finite E_∞. For example, $x(t) = e^{-|t|}$, and, trivially, $x(t) = 0$, for all *t* are energy signals. For an energy signal, $P_\infty = 0$.
- A *power signal* is a signal with finite, nonzero P_∞. An example is $x(t) = 1$, for all *t*, though more interesting examples are not obvious and require analysis. For a power signal, $E_\infty = \infty$.

Example Most would suspect that $x(t) = \sin(t)$ is not an energy signal, but in any case we first compute

$$\int_{-T}^{T} \sin^2(t)\, dt = \int_{-T}^{T} \left(\tfrac{1}{2} - \tfrac{1}{2}\cos(2t)\right) dt = T - \tfrac{1}{2}\sin(2T)$$

Letting $T \to \infty$ confirms our suspicions, since the limit doesn't exist. The second step of the power-signal calculation gives

$$P_\infty = \lim_{T\to\infty} \tfrac{1}{2T}\left(T - \tfrac{1}{2}\sin(2T)\right) = \tfrac{1}{2}$$

and we conclude that $x(t)$ is a power signal.

Example The unit-step function, defined by

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

is a power signal, since

$$\begin{aligned} \lim_{T\to\infty} \tfrac{1}{T} \int_{-T/2}^{T/2} u^2(t)\, dt &= \lim_{T\to\infty} \tfrac{1}{T} \int_{0}^{T/2} 1\, dt \\ &= \lim_{T\to\infty} \tfrac{1}{T}\tfrac{T}{2} = \tfrac{1}{2} \end{aligned}$$

Example There are signals that belong to neither of these classes. For example, $x(t) = e^t$ is a signal with both E_∞ and P_∞ infinite. A more unusual example is

$$x(t) = \begin{cases} t^{-1/2}, & t \geq 1 \\ 0, & t < 1 \end{cases}$$

This signal has infinite energy but zero average power.

The *RMS* (*root-mean-square*) value of a power signal $x(t)$ is defined as $\sqrt{P_\infty}$.
These energy and power definitions also can be used for complex-valued signals, in which case we replace $x^2(t)$ by $|x(t)|^2$.

1.5 Symmetry-Based Classifications of Signals

A signal $x(t)$ is called an *even signal* if $x(-t) = x(t)$ for all *t*. If $x(-t) = -x(t)$, for all *t*, then $x(t)$ is called an *odd signal*.

The *even part* of a signal $x(t)$ is defined as

$$x_{ev}(t) = \frac{x(t) + x(-t)}{2}$$

and the *odd part* of $x(t)$ is

$$x_{od}(t) = \frac{x(t) - x(-t)}{2}$$

The even part of a signal is an even signal, since

$$x_{ev}(-t) = \frac{x(-t) + x(t)}{2} = x_{ev}(t)$$

and a similar calculation shows that the odd part of a signal is an odd signal. Also, for any signal $x(t)$ we can write a decomposition as

$$x(t) = x_{ev}(t) + x_{od}(t)$$

These concepts are most useful for real signals. For complex-valued signals, a symmetry concept that sometimes arises is *conjugate symmetry*, characterized by

$$x(t) = x^*(-t)$$

where superscript star denotes complex conjugate.

1.6 Additional Classifications of Signals

- *Boundedness*: A signal $x(t)$ is called *bounded* if there is a finite constant *K* such that $|x(t)| \le K$, for all *t*. (Here the absolute value is interpreted as magnitude if the signal is complex valued.) Otherwise a signal is called *unbounded*. That is, a signal is unbounded if no such *K* exists. For example, $x(t) = \sin(3t)$ is a bounded signal, and we can take $K = 1$. Obviously, $x(t) = t\sin(3t)$ is unbounded.

- *Periodicity*: A signal $x(t)$ is called *periodic* if there is a positive constant *T* such that $x(t) = x(t+T)$, for all *t*. Such a *T* is called a *period* of the signal, and sometimes we say a signal is *T-periodic*. Of course if a periodic signal has period *T*, then it also has period *2T*, *3T*, and so on. The smallest value of *T* for which $x(t) = x(t+T)$, for all *t*, is called the *fundamental period* of the signal, and often is denoted T_o. Note also that a constant signal, *x(t) = 3*, for example, is periodic with period any *T > 0*, and the fundamental period is not well defined (there is no smallest positive number).

Examples To determine periodicity of the signal $x(t) = \sin(3t)$, and the fundamental period T_o if periodic, we apply the periodicity condition

$$\sin(3(t+T) = \sin(3t)\,, \quad -\infty < t < \infty$$

Rewriting this as

$$\sin(3t+3T) = \sin(3t)\,, \quad -\infty < t < \infty$$

it is clear that the condition holds if and only if $3T$ is an integer multiple of 2π, that is, T is a positive integer multiple of $2\pi/3$. Thus the signal is periodic, and the fundamental period is $T_o = 2\pi/3$. As a second example, we regard $x(t) = u(t) + u(-t)$ as periodic, by assuming for convenience the value $u(0) = 1/2$, but there is no fundamental period.

Periodic signals are an important subclass of all signals. Physical examples include the ocean tides, an at-rest ECG, and musical tones (but not tunes).

Typically we consider the period of a periodic signal to have units of *seconds*, and the *fundamental frequency* of a periodic signal is defined by

$$\omega_o = \frac{2\pi}{T_o}$$

with units of *radians/second*. We will use radian frequency throughout, though some other sources use frequency in *Hertz*, denoted by the symbol f_o. The relation between radian frequency and Hertz is

$$f_o = \frac{\omega_o}{2\pi} = \frac{1}{T_o}$$

The main difference that arises between the use of the two frequency units involves the placement of 2π factors in various formulas.

Given a literal expression for a signal, solving the equation

$$x(t) = x(t+T), \quad \textit{for all } t$$

for the smallest value of T, if one exists, can be arbitrarily difficult. Sometimes the best approach is to plot out the signal and try to determine periodicity and the fundamental period by inspection. Such a conclusion is not definitive, however, since there are signals that are very close to, but not, periodic, and this cannot be discerned from a sketch.

Note that the average power per period of a T-periodic signal $x(t)$ becomes,

$$P_T = \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)\, dt$$

or, more generally,

$$P_T = \frac{1}{T}\int_{T} x^2(t)\, dt$$

where we have indicated that the integration can be performed over any interval of length T. To prove this, for any constant t_o consider

$$\frac{1}{T}\int_{t_o}^{t_o+T} x^2(t)\, dt$$

and perform the variable change $\tau = t - t_o - T/2$. The average power per period is the same as the average power of the periodic signal. Therefore the RMS value of a periodic signal $x(t)$ is

$$\sqrt{P_\infty} = \left(\frac{1}{T} \int_T x^2(t)\, dt \right)^{\frac{1}{2}}$$

Example Ordinary household electrical power is supplied as a 60 Hertz sinusoid with RMS value about 110 volts. That is,

$$x(t) = A\cos(120\pi t)$$

and the fundamental period is $T_o = 1/60$ sec. The amplitude *A* is such that

$$110 = \left(60 \int_0^{1/60} A^2 \cos^2(120\pi t)\, dt \right)$$

from which we compute $A \approx 140$.

1.7 Discrete-Time Signals: Definitions, Classifications, and Operations

For discrete-time signals, $x[n]$, we simply need to convert the various notions from the setting of functions to the setting of sequences.

- *Energy and Power*: The *total energy* of a discrete-time signal is defined by

$$E_\infty = \sum_{n=-\infty}^{\infty} x^2[n] = \lim_{N\to\infty} \sum_{n=-N}^{N} x^2[n]$$

The time-average power is

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x^2[n]$$

and discrete-time classifications of energy signals and power signals are defined exactly as in the continuous-time case.

Examples The unit pulse signal,

$$\delta[n] = \begin{cases} 1\,, & n = 0 \\ 0\,, & n \neq 0 \end{cases}$$

is an energy signal, with $E_\infty = 1$. The unit-step signal,

$$u[n] = \begin{cases} 1\,, & n \geq 0 \\ 0\,, & n < 0 \end{cases}$$

is a power signal with time-average power

$$
\begin{aligned}
P_\infty &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} u^2[n] \\
&= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} 1 \\
&= \lim_{N\to\infty} \frac{N+1}{2N+1} = \frac{1}{2}
\end{aligned}
$$

- *Periodicity*: The signal $x[n]$ is *periodic* if there is a positive integer *N*, called a *period*, such that

$$x[n+N] = x[n]$$

for all integer *n*. The smallest period of a signal, that is, the least value of *N* such that the periodicity condition is satisfied, is called the *fundamental period* of the signal. The fundamental period is denoted N_o, though sometimes the subscript is dropped in particular contexts.

Example To check periodicity of the signal $x[n] = \sin(3n)$, we check if there is a positive integer *N* such that

$$\sin(3(n+N)) = \sin(3n)\,, \quad n = 0, \pm 1, \pm 2, \ldots$$

That is

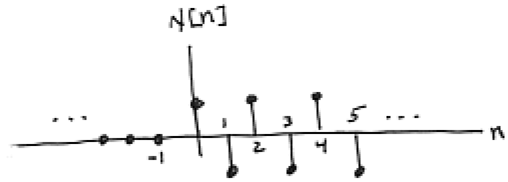
$$\sin(3n+3N) = \sin(3n)\,, \quad n = 0, \pm 1, \pm 2, \ldots$$

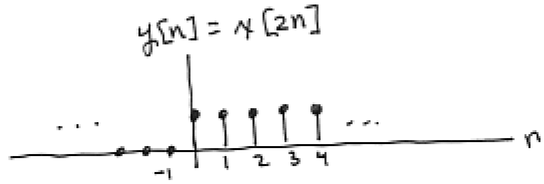
This condition holds if and only if *3N* is an integer multiple of 2π, a condition that cannot be met by integer *N*. Thus the signal is not periodic.

- *Elementary operations*: Elementary operations, for example addition and scalar multiplication, on discrete-time signals are obvious conversions from the continuous-time case. Elementary transformations of the independent variable also are easy, though it must be remembered that only integer argument values are permitted in the discrete-time case

- *Time Scale*: Suppose $y[n] = x[an]$, where *a* is a positive or negative integer (so that the product, *an*, is an integer for all integer *n*). If $a = -1$, this is a time reversal. But for any case beyond $a = \pm 1$, be aware that loss of information in the signal occurs, unlike the continuous-time case.

Example For $a = 2$, compare the time scaled signal with the original:





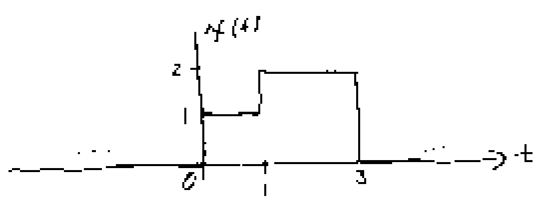
- *Time Shift*: Suppose $y[n] = x[n - N]$, where *N* is a fixed integer. If *N* is positive, then this is a right shift, or delay, and if *N* is negative, it is a left shift or advance.

- *Combination Scale and Shift*: Suppose $y[n] = x[an - N]$, where *a* is a nonzero integer and *N* is an integer. As in the continuous-time case, the safest approach to interpreting the result is to simply plot out the signal *y*[*n*].

Example: Suppose $y[n] = x[N - n]$. This is a *flip and shift*, and occurs sufficiently often that it is worthwhile verifying and remembering the shortcut: *y*[*n*] can be plotted by time-reversing (flipping) *x*[*n*] and then shifting the reversed signal to move the original value at $n = 0$ to $n = N$. That is, shift *N* samples to the right if $N > 0$, and $|N|$ samples to the left if $N < 0$.

Exercises

1. Given the signal shown below,



sketch the signal $y(t) =$

(a) $x(t) - x(t-1)$

(b) $x(-2t)$

(c) $x(t-1)u(1-t)$

(d) $x(2t) + x(-3t)$

(e) $x(3t-1)$

2. Determine if the following signals are power signals or energy signals, and compute the total energy or time-average power, as appropriate.

(a) $x(t) = \sin(2t)u(t)$

(b) $x(t) = e^{-|t|}$

(c) $x(t) = tu(t)$

(d) $x(t) = 5e^{-3t}u(t)$

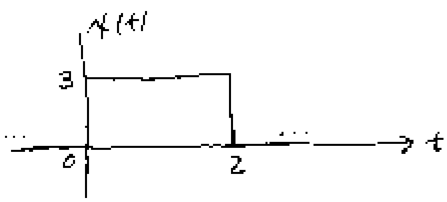
3. For an energy signal $x(t)$, prove that the total energy is the sum of the total energy of the even part of $x(t)$ and the total energy of the odd part of $x(t)$.

4. If a given signal $x(t)$ has total energy $E = 5$, what is the total energy of the signal $y(t) = 2x(3t-4)$?

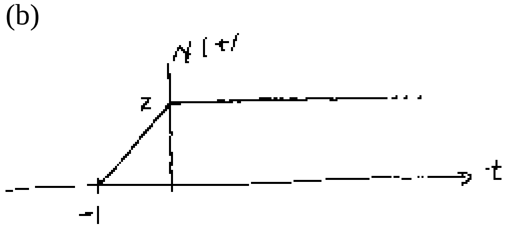
5. Under what conditions on the real constant α is the continuous-time signal $x(t) = e^{\alpha t}u(-t)$ an energy signal? When your conditions are satisfied, what is the energy of the signal?

6. Sketch the even and odd parts of the signals below.

(a)



(b)



7. Suppose that for a signal $x(t)$ it is known that $Ev\{x(t)\} = Od\{x(t)\} = 1$ for $t > 0$. What is $x(t)$?

8. Determine which of the following signals are periodic, and specify the fundamental period.

(a) $x(t) = e^{j\pi}\cos(2\pi t + \pi)$

(b) $x(t) = \sin^2(t)$

(c) $x(t) = \sum_{k=-\infty}^{\infty} u(t-2k) - u(t-1-2k)$

(d) $x(t) = 3e^{2-j3\pi t}$

9. Suppose that $x_1(t)$ and $x_2(t)$ are periodic signals with respective fundamental periods T_1 and T_2. Show that if there are positive integers *m* and *n* such that

$$\frac{T_1}{T_2} = \frac{m}{n}$$

(that is, the ratio of fundamental periods is a rational number), then $x(t) = x_1(t) + x_2(t)$ is periodic. If the condition holds, what is the fundamental period of $x(t)$?

10. Determine which of the following signals are bounded, and specify a smallest bound.

(a) $x(t) = e^{3t}u(t)$

(b) $x(t) = e^{3t}u(-t)$

(c) $x(t) = 4e^{-6|t|}$

(d) $x(t) = -2\,t\,e^{-3t}\sin(t^5)\,u(t)$

11. Given the signal $x[n] = \delta[n] - \delta[n-1]$, sketch $y[n] =$

(a) $x[4n-1]$

(b) $x[n]u[1-n]$

(c) $3x[-2n+3]$

(d) $x[2n] - x[1-n]$

12. Determine whether the following signals are periodic, and if so determine the fundamental period.

(a) $x[n] = u[n] + u[-n]$

(b) $x[n] = e^{-j3\pi n}$

(c) $x[n] = (-1)^n + e^{j\frac{\pi}{2}n}$

(d) $x[n] = \cos(\frac{\pi}{4}n)$

13. Suppose *x[n]* is a discrete-time signal, and let *y[n]=x[2n]*.

(a) If *x[n]* is periodic, is *y[n]* periodic? If so, what is the fundamental period of *y[n]* in terms of the fundamental period of x*[n]*?

(b) If *y[n]* is periodic, is *x[n]* periodic? If so, what is the fundamental period of *x[n]* in terms of the fundamental period of *y[n]*?

14. Under what condition is the sum of two periodic discrete-time signals periodic? When the condition is satisfied, what is the fundamental period of the sum, in terms of the fundamental periods of the summands?

15. Is the signal $x[n] = 3(-1)^n u[n]$ an energy signal, power signal, or neither?

16. Is the signal $x[n] = e^{-j2\pi n} + e^{j\pi n}$ periodic? If so, what is the fundamental period?

17. Answer the following questions about the discrete-time signal $x[n] = e^{-j(\pi/2)n}$.

(a) Is $x[n]$ periodic? If so, what is its fundamental period?

(b) Is $x[n]$ an even signal? Is it an odd signal?

(c) Is $x[n]$ an energy signal? Is it a power signal?

18. Which of the following signals are periodic? For those that are periodic, what is the fundamental period?

(a) $x[n] = e^{j\frac{4}{\pi}n}$

(b) $x[n] = e^{j\frac{2}{8}\pi n}$

(c) $x[n] = e^{-j\frac{7}{8}\pi(n-1)}$

Notes for Signals and Systems

Much of our discussion will focus on two broad classes of signals: the class of complex exponential signals and the class of singularity signals. Though it is far from obvious, it turns out that essentially all signals of interest can be addressed in terms of these two classes.

2.1 The Class of CT Exponential Signals

There are several ways to represent the complex-valued signal

$$x(t) = c\,e^{at}\ , \quad -\infty < t < \infty$$

where both c and a are complex numbers. A convenient approach is to write c in polar form, and a in rectangular form,

$$c = |c|e^{j\phi_o} \quad , \quad a = \sigma_o + j\omega_o$$

where $\phi_o = \angle c$ and where we have chosen notations for the rectangular form of a that are customary in the field of signals and systems. In addition, the subscript o's are intended to emphasize that the quantities are fixed real numbers. Then

$$\begin{aligned} x(t) &= |c|e^{j\phi_o}\ e^{(\sigma_o + j\omega_o)t} \\ &= |c|e^{\sigma_o t}\ e^{j(\omega_o t + \phi_o)} \end{aligned}$$

Using Euler's formula, we can write the signal in rectangular form as

$$x(t) = |c|e^{\sigma_o t}\cos(\omega_o t + \phi_o) + j\,|c|e^{\sigma_o t}\sin(\omega_o t + \phi_o)$$

There are two special cases that are of most interest.

Special Case 1: Suppose both c and a are real. That is, $\omega_o = 0$ and ϕ_o is either *0* or π. Then we have the familiar exponentials

$$x(t) = \begin{cases} |c|e^{\sigma_o t}\ , & if\ \phi_o = 0 \\ -|c|e^{\sigma_o t}\ , & if\ \phi_o = \pi \end{cases}$$

Or, more simply,

$$x(t) = c\,e^{\sigma_o t}$$

Special Case 2: Suppose c is complex and a is purely imaginary. That is, $\sigma_o = 0$. Then

$$\begin{aligned} x(t) &= |c|e^{j(\omega_o t + \phi_o)} \\ &= |c|\cos(\omega_o t + \phi_o) + j\,|c|\sin(\omega_o t + \phi_o) \end{aligned}$$

Both the real and imaginary parts of $x(t)$ are periodic signals, with fundamental period

$$T_o = \frac{2\pi}{|\omega_o|}$$

Since the independent variable, t, is viewed as time, units of ω_o typically are *radians/second* and units of T_o naturally are seconds. A signal of this form is often called a *phasor*.

Left in exponential form, we can check directly that given any ω_o, $x(t)$ is periodic with period $T_o = 2\pi / |\omega_o|$:

$$\begin{aligned} x(t+T_o) &= |c| e^{j[\omega_o(t+T_o)+\phi_o]} \\ &= |c| e^{j(\omega_o t+\phi_o)} e^{\pm j2\pi} \\ &= |c| e^{j(\omega_o t+\phi_o)} \\ &= x(t) \end{aligned}$$

Also, it is clear that T_o is the fundamental period of the signal, simply by attempting to satisfy the periodicity with any smaller, positive value for the period.

We can view a phasor signal as a vector at the origin of length $|c|$ rotating in the complex plane with angular frequency ω_o radians/second, beginning with the angle ϕ_o at $t = 0$. If $\omega_o > 0$, then the rotation is counter clockwise. If $\omega_o < 0$, then the rotation is clockwise. Of course, if $\omega_o = 0$, then the signal is a constant, and it is not surprising that the notion of a fundamental period falls apart.

The applet in the link below illustrates this rotating-vector interpretation, and also displays the imaginary part of the phasor, that is, the projection on the vertical axis.

One Phasor

Sums of phasors that have different frequencies are also very important. These are best visualized using the "head-to-tail" construction of vector addition. The applet below illustrates.

Sum of Two Phasors

The question of periodicity becomes much more interesting for phasor sums, and we first discuss this for sums of two phasors. Consider

$$x(t) = c_1 e^{j\omega_1 t} + c_2 e^{j\omega_2 t}$$

The values of c_1 and c_2 are not essential factors in the periodicity question, but the values of ω_1 and ω_2 are. It is worthwhile to provide a formal statement and proof of the result, with assumptions designed to rule out trivialities and needless complexities.

Theorem The complex valued signal

$$x(t) = c_1 e^{j\omega_1 t} + c_2 e^{j\omega_2 t}$$

with $c_1, c_2 \neq 0$ and $\omega_2, \omega_1 \neq 0$ is periodic if and only if there exists a positive frequency ω_0 and integers *k* and *l* such that

$$\omega_1 = k\omega_0 , \quad \omega_2 = l\omega_0 \qquad (2.1)$$

Furthermore, if ω_0 is the largest frequency for which (2.1) can be satisfied, in which case it is called the *fundamental frequency* for $x(t)$, then the fundamental period of $x(t)$ is $T_o = 2\pi / \omega_0$.

Proof First we assume that positive ω_0 and the integers *k, l* satisfy (2.1). Choosing $T = 2\pi / \omega_0$, we see that

$$\begin{aligned} x(t+T) &= c_1 e^{jk\omega_0(t+T)} + c_2 e^{jl\omega_0(t+T)} \\ &= e^{jk2\pi} c_1 e^{jk\omega_0 t} + e^{jl2\pi} c_2 e^{jl\omega_0 t} \\ &= c_1 e^{jk\omega_0 t} + c_2 e^{jl\omega_0 t} \\ &= x(t) \end{aligned}$$

for all *t*, and $x(t)$ is periodic. It is easy to see that if ω_o is the largest frequency such that (2.1) is satisfied, then the fundamental period of $x(t)$ is $T_o = 2\pi / \omega_o$. (Perhaps we are beginning to abuse the subscript oh's and zeros…)

Now suppose that *x(t)* is periodic, and $T > 0$ is such that $x(t+T) = x(t)$ for all *t*. That is,

$$c_1 e^{j\omega_1(t+T)} + c_2 e^{j\omega_2(t+T)} = c_1 e^{j\omega_1 t} + c_2 e^{j\omega_2 t}$$

for all *t*. This implies

$$(e^{j\omega_1 T} - 1)c_1 + (e^{j\omega_2 T} - 1)\, c_2 e^{j(\omega_2 - \omega_1)t} = 0$$

for all *t*. Picking the particular times $t = 0$ and $t = \pi / (\omega_2 - \omega_1)$ gives the two algebraic equations

$$(e^{j\omega_1 T} - 1)c_1 + (e^{j\omega_2 T} - 1)\, c_2 = 0$$

$$(e^{j\omega_1 T} - 1)c_1 - (e^{j\omega_2 T} - 1)\, c_2 = 0$$

By adding these two equations, and also subtracting the second from the first, we obtain

$$e^{j\omega_1 T} = e^{j\omega_2 T} = 1$$

Therefore both frequencies must be integer multiples of frequency $2\pi / T$.

Example The signal

$$x(t) = 4e^{j2t} - 5e^{j3t}$$

is periodic with fundamental frequency $\omega_o = 1$, and thus fundamental period 2π. The signal

$$x(t) = 4e^{j2t} - 5e^{j\pi t}$$

is not periodic, since the frequencies *2* and π cannot be integer multiples of a fixed frequency.

The theorem generalizes to the case where $x(t)$ is a sum of any number of complex exponential terms: the signal is periodic if and only if there exists ω_0 such that every frequency present in the sum can be written as an integer multiple of ω_0. Such frequency terms are often called *harmonically related*. The applet below can be used to visualize sums of several harmonically related phasors, and the imaginary part exhibits the corresponding periodic, real signal.

Phasor Sums

2.2 The Class of CT Singularity Signals

The basic singularity signal is the *unit impulse*, $\delta(t)$, a signal we invent in order to have the following *sifting property* with respect to ordinary signals, $x(t)$:

$$\int_{-\infty}^{\infty} x(t)\delta(t)\, dt = x(0) \tag{2.2}$$

That is, $\delta(t)$ causes the integral to "sift out" the value of $x(0)$. Here $x(t)$ is any continuous-time signal that is a continuous function at *t = 0*, so that the value of $x(t)$ at $t = 0$ is well defined. For example, a unit step, or the signal $x(t) = 1/t$, would not be eligible for use in the sifting property. (However, some treatments do allow a finite jump in $x(t)$ at $t = 0$, as occurs in the unit step signal, and the sifting property is defined to give the mid-point of the jump. That is,

$$\int_{-\infty}^{\infty} x(t)\delta(t)\, dt = \frac{x(0^+) + x(0^-)}{2}$$

For example, if the signal is the unit step, then the sift would yield $1/2$.)

A little thought, reviewed in detail below, shows that $\delta(t)$ cannot be a function in the ordinary sense. However, we develop further properties of the unit impulse by focusing on implications of the sifting property, while insisting that in other respects $\delta(t)$ behave in a manner consistent with the usual rules of arithmetic and calculus of ordinary functions.

- *Area*

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{2.3}$$

Considering the sifting property with the signal $x(t) = 1$, for all *t*, we see the unit impulse must satisfy (2.3).

- *Time values*

$$\delta(t) = 0, \ for\ t \neq 0 \tag{2.4}$$

By considering $x(t)$ to be any signal that is continuous at $t = 0$ with $x(0) = 0$, for example, the signals $x(t) = t, t^2, t^3, \ldots$, it can be shown that there is no contribution to the integral in (2.2) for nonzero values of the integration variable. This indicates that the impulse must be zero for nonzero arguments. Obviously $\delta(0)$ cannot be zero, and indeed it must have, in some sense, infinite value. That is, the unit impulse is zero everywhere except $t = 0$, and yet has unit area. This makes clear the fact that we are dealing with something outside the realm of basic calculus.

Notice also that these first two properties imply that

$$\int_{-a}^{a} \delta(t)\, dt = 1$$

for any $a > 0$.

- *Scalar multiplication*

We treat the scalar multiplication of an impulse the same as the scalar multiplication of an ordinary signal. To interpret the sifting property for $a\,\delta(t)$, where *a* is a constant, note that the properties of integration imply

$$\int_{-\infty}^{\infty} x(t)\,[a\delta(t)]\,dt = a\int_{-\infty}^{\infty} x(t)\,\delta(t)\,dt = a\,x(0)$$

The usual terminology is that $a\delta(t)$ is an "impulse of area *a*," based on choosing $x(t)=1$, for all *t*, in the sifting expression.

- *Signal Multiplication*

$$z(t)\delta(t) = z(0)\delta(t)$$

When a unit impulse is multiplied by a signal *z*(*t*), which is assumed to be continuous at $t = 0$, the sifting property gives

$$\int_{-\infty}^{\infty} x(t)\,[z(t)\delta(t)]\,dt = \int_{-\infty}^{\infty} [x(t)z(t)]\,\delta(t)\,dt = x(0)z(0)$$

This is the same as the result obtained when the unit impulse is multiplied by the constant $z(0)$,

$$\int_{-\infty}^{\infty} x(t)[z(0)\delta(t)]\,dt = z(0)\int_{-\infty}^{\infty} x(t)\delta(t)\,dt = z(0)x(0)$$

Therefore we conclude the signal multiplication property shown above.

- *Time shift*

We treat the time shift of an impulse the same as the time shift of any other signal. To interpret the sifting property for the time shifted unit impulse, $\delta(t-t_o)$, a change of integration variable from *t* to $\tau = t - t_o$ gives

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-t_o)\,dt = \int_{-\infty}^{\infty} x(\tau+t_o)\,\delta(\tau)\,d\tau = x(t_o)$$

This property, together with the function multiplication property gives the more general statement

$$z(t)\delta(t-t_o) = z(t_o)\delta(t-t_o)$$

where t_o is any real constant and *z*(*t*) is any ordinary signal that is a continuous function of *t* at $t = t_o$.

- *Time scale*

Since an impulse is zero for all nonzero arguments, time scaling an impulse has impact only with regard to the sifting property where, for any nonzero constant *a*,

$$\int_{-\infty}^{\infty} x(t)\,\delta(at)\,dt = \frac{1}{|a|}x(0), \quad a \neq 0$$

To justify this expression, assume first that $a > 0$. Then the sifting property must obey, by the principle of consistency with the usual rules of integration, and in particular with the change of integration variable from t to $\tau = at$,

$$\int_{-\infty}^{\infty} x(t)\,\delta(at)\,dt = \frac{1}{a}\int_{-\infty}^{\infty} x(\tau / a)\,\delta(\tau)\,d\tau = \frac{1}{a}x(0)\,, \quad a > 0$$

A similar calculation for $a < 0$, where now the change of integration variable yields an interchange of limits, gives

$$\int_{-\infty}^{\infty} x(t)\,\delta(at)\,dt = \frac{1}{a}\int_{\infty}^{-\infty} x(\tau / a)\,\delta(\tau)\,d\tau$$

$$= -\frac{1}{a}\int_{-\infty}^{\infty} x(\tau / a)\,\delta(\tau)\,d\tau = -\frac{1}{a}x(0)\,, \quad a < 0$$

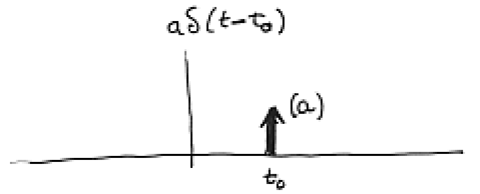
These two cases can be combined into one expression given above. Thus the sifting property leads to the definition:

$$\delta(at) = \frac{1}{|\,a\,|}\,\delta(t)\,, \quad a \neq 0$$

- *Symmetry*

Note that the case $a = 1$ in time scaling gives the result that $\delta(-t)$ acts in the sifting property exactly as $\delta(t)$, so we regard the unit impulse as an "even function." Other interpretations are possible, but we will not go there.

We graphically represent an impulse by an arrow, as shown below.



(If the area of the impulse is negative, $a < 0$, sometimes the arrow is drawn pointing south.)

We could continue this investigation of properties of the impulse, for example, using the calculus consistency principle to figure out how to interpret $\delta(at - t_o)$, $z(t)\delta(at)$, and so on. But we only need the properties justified above, and two additional properties that are simply wondrous. These include an extension of the sifting property that violates the continuity condition:

- *Special Property 1*

$$\int_{-\infty}^{\infty} \delta(\tau)\delta(t-\tau)\,d\tau = \delta(t)$$

Note here that the integration variable is τ, and t is any real value. Even more remarkable is an expression that relates impulses and complex exponentials:

- *Special Property 2*

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{jt\omega}\,d\omega$$

Note here that the integral simply does not converge in the usual sense of basic calculus, since $|e^{jt\omega}|=1$ for any (real) values of t and ω.

Remark Our general approach to these impulse properties will be "don't think about impulses… simply follow the rules." However, to provide a bit of explanation, with little rigor, we briefly discuss one of the mathematical approaches to the subject. To arrive at the unit impulse, consider the possibility of an infinite sequence of functions, $d_n(t), \quad n=1,2,3,\ldots$, that have the unit-area property

$$\int_{-\infty}^{\infty} d_n(t)\,dt = 1, \quad n=1,2,3,\ldots$$

and also have the property that for any other function $x(t)$ that is continuous at $t = 0$,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} d_n(t)x(t)\,dt = x(0)$$

Here the limit involves a sequence of numbers defined by ordinary integrals, and can be interpreted in the usual way. However we next interchange the order of the limit and the integration, without proper justification, and view $\delta(t)$ as "some sort" of limit:

$$\delta(t) = \lim_{n\to\infty} d_n(t)$$

This view is useful for intuition purposes, but is dangerous if pursued too far by elementary means. In particular, for the sequences of functions $d_n(t)$ typically considered, the limit does not exist in any usual sense.

Examples Consider the rectangular-pulse signals

$$d_n(t) = \begin{cases} n, & \frac{-1}{2n} < t < \frac{1}{2n} \\ 0, & else \end{cases} \quad , \quad n=1,2,3,\ldots$$

The pulses get taller and thinner as n increases, but clearly every $d_n(t)$ is unit area, and the mean-value theorem can be used to show

$$\int_{-\infty}^{\infty} d_n(t)x(t)\,dt = n \int_{-1/(2n)}^{1/(2n)} x(t)\,dt \approx n\frac{x(0)}{n}$$

with the approximation getting better as n increases. Thus we can casually view a unit impulse as the limit, as $n\to\infty$, of these unit-area rectangles. A similar example is to take $d_n(t)$ to be a triangle of height n, width $2/n$, centered at the origin. But it turns out that a more interesting example is to use the *sinc* function defined by

$$\text{sinc}(t) = \frac{\sin(\pi t)}{(\pi t)}$$

and let

$$d_n(t) = n\sin c(nt) \quad , \quad n=1,2,3,\ldots$$

It can be shown, by evaluating an integral that is not quite elementary, that these signals all have area 2π, and that the sifting property

$$\int_{-\infty}^{\infty} d_n(t)x(t)\,dt \approx x(0)$$

is a better and better approximation as n grows without bound. Therefore we can view an impulse of area 2π, that is, $2\pi\delta(t)$, as a limit of these functions. This sequence of *sinc* signals is displayed in the applet below for a range of n, and you can get a pictorial view of how an impulse might arise from *sinc's* as n increases, in much the same way as the impulse arises from height n, width $1/n$, rectangular pulses as n increases.

Family of Sincs

Remark Special Property 2 can be intuitively understood in terms of our casual view of impulses as follows. Let

$$\begin{aligned} d_W(t) &= \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\,d\omega \\ &= \frac{1}{2\pi}\int_{-W}^{W} [\cos(\omega t) + j\sin(\omega t)]\,d\omega \\ &= \frac{1}{2\pi}\int_{-W}^{W} \cos(\omega t)\,d\omega + \frac{j}{2\pi}\int_{-W}^{W} \sin(\omega t)\,d\omega \end{aligned}$$

Using the fact that a sinusoid is an odd function of its argument,

$$\begin{aligned} d_W(t) &= \frac{1}{\pi}\int_{0}^{W} \cos(\omega t)\,d\omega \\ &= \frac{1}{\pi}\frac{\sin(Wt)}{t} \\ &= \frac{W}{\pi}\,\mathrm{sinc}\left(\frac{Wt}{\pi}\right) \end{aligned}$$

This $d_W(t)$ can be shown to have unit area for every $W > 0$, again by a non-elementary integration, and again the sifting property is approximated when W is large. Therefore the Special Property 2 might be expected. The applet below shows a plot of $d_W(t)$ as W is varied, and provides a picture of how the impulse might arise as W increases.

Another Sinc Family

• *Additional Singularity Signals*

From the unit impulse we generate additional singularity signals using a generalized form of calculus. Integration leads to

$$\int_{-\infty}^{t} \delta(\tau)\,d\tau = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

which is the familiar *unit-step function*, $u(t)$. (We leave the value of $u(0)$, where the jump occurs, freely assignable following our general policy.)
The "running integral" in this expression actually can be interpreted graphically in very nice way. And a variable change from τ to $\sigma = t - \tau$ gives the alternate expression

$$u(t) = \int_0^{\infty} \delta(t-\sigma)\, d\sigma$$

Analytically this can be viewed as an application of a sifting property applied to the case $x(t) = u(t)$:

$$\int_0^{\infty} \delta(t-\sigma)\, d\sigma = \int_{-\infty}^{\infty} u(\sigma)\delta(t-\sigma)\, d\sigma = \int_{-\infty}^{\infty} u(t-\sigma)\delta(\sigma)\, d\sigma = u(t)$$

This is not, strictly speaking, legal for $t = 0$, because of the discontinuity there in $u(t)$, but we ignore this issue.

By considering the running integral of the unit-step function, we arrive at the *unit-ramp*:

$$\int_{-\infty}^{t} u(\tau)\, d\tau = \begin{cases} 0, & t < 0 \\ t, & t \ge 0 \end{cases}$$
$$= tu(t)$$

Often we write this as $r(t)$. Notice that the unit ramp is a continuous function of time, though it is unbounded.
Continuing,

$$\int_{-\infty}^{t} r(\tau)\, d\tau = \begin{cases} 0, & t < 0 \\ t^2/2, & t \ge 0 \end{cases}$$
$$= \frac{t^2}{2} u(t)$$

which might be called the *unit parabola*, $p(t)$. We stop here, as further integrations yield signals little used in the sequel.

We can turn matters around, using differentiation and the fundamental theorem of calculus. Clearly,

$$\frac{d}{dt}\, p(t) = \frac{d}{dt} \int_{-\infty}^{t} r(\tau)\, d\tau = r(t)$$

and this is a perfectly legal application of the fundamental theorem since the integrand, $r(t)$, is a continuous function of time. However, we go further, cheating a bit on the assumptions, since the unit step is not continuous, to write

$$\frac{d}{dt} r(t) = \frac{d}{dt} \int_{-\infty}^{t} u(\tau)\, d\tau = u(t)$$

That this cheating is not unreasonable follows from a plot of the unit ramp, $r(t)$, and then a plot of the slope at each value of t.
Cheating more, we also write

$$\frac{d}{dt} u(t) = \frac{d}{dt} \int_{-\infty}^{t} \delta(\tau)\, d\tau = \delta(t)$$

Again, a graphical interpretation makes this seem less unreasonable.

We can also consider “derivatives” of the unit impulse. The approach is again to demand consistency with other rules of calculus, and use integration by parts to interpret the “sifting property” that should be satisfied. We need go no further than the first derivative, where

$$\int_{-\infty}^{\infty} x(t)[\tfrac{d}{dt}\,\delta(t)]\,dt = x(t)\,\delta(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \dot{x}(t)\,\delta(t)\,dt$$

$$= -\dot{x}(0)$$

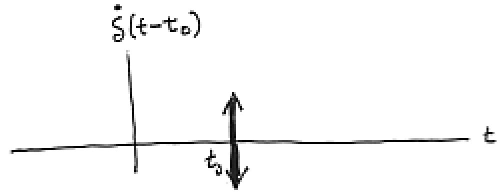
This assumes, of course, that $\dot{x}(t)$ is continuous at $t = 0$, that is, $x(t)$ is continuously differentiable at $t = 0$. This unit-impulse derivative is usually called the *unit doublet*, and denoted $\dot{\delta}(t)$. Various properties can be deduced, just as for the unit impulse. For example, choosing $x(t) = 1,\ -\infty < t < \infty,$ the sifting property for the doublet gives

$$\int_{-\infty}^{\infty} \dot{\delta}(t)\,dt = 0$$

In other words, the doublet has zero area – a true ghost. It is also easy to verify the property

$$\int_{-\infty}^{\infty} x(t)\,\dot{\delta}(t-t_o)\,dt = -\,\dot{x}(t_o)$$

and, finally, we sketch the unit doublet as shown below.



All of the "generalized calculus" properties can be generalized in various ways. For example, the product rule gives

$$\frac{d}{dt}\left[t\,u(t)\right] = 1\,u(t) + t\,\delta(t)$$

$$= u(t)$$

where we have used the multiplication rule to conclude that $t\delta(t) = 0$ for all t. As another example, the chain rule gives

$$\frac{d}{dt}\,u(t-t_o) = \delta(t-t_o)$$

Remark These matters can be taken too far, to a point where ambiguities begin to overwhelm and worrisome liberties must be taken. For example, using the product rule for differentiation, and ignoring the fact that $u^2(t)$ is the same signal as $u(t)$,

$$\tfrac{d}{dt}\,u^2(t) = \dot{u}(t)u(t) + u(t)\dot{u}(t) = 2u(t)\delta(t)$$

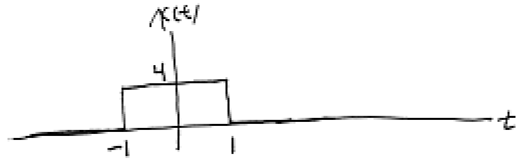
The multiplication rule for impulses does not apply, since $u(t)$ is not continuous at $t = 0$, and so we are stuck. However if we interpret $u(0)$ as ½, the midpoint of the jump, we get a result consistent with $\dot{u}(t) = \delta(t)$. We will not need to take matters this far, since we use these generalized notions only for rather simple signals.

2.3 Linear Combinations of Singularity Signals and Generalized Calculus

For simple signals, that is, signals with uncomplicated wave shapes, it is convenient for many purposes to use singularity signals for representation and calculation.

Example The signal shown below can be written as a sum of step functions,

$$x(t) = 4u(t+1) - 4u(t-1)$$



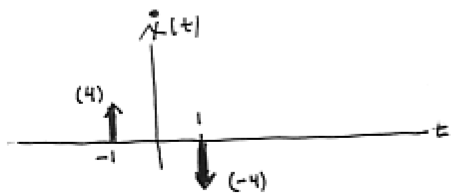
Another representation is

$$x(t) = 4u(t+1)u(1-t)$$

This uses the unit steps as “cutoff” functions, and sometimes this usage is advantageous. However for further calculation, representation as a linear combination usually is much simpler. Differentiation of the first expression for $x(t)$ gives, using our generalized notion of differentiation,

$$\dot{x}(t) = 4\delta(t+1) - 4\delta(t-1)$$

This signal is shown below.



The same result can be obtained by differentiating the “step cutoff” representation for *x*(*t*), though usage of the product rule and interpretation of the final result makes the derivative calculation more difficult. That is,

$$\begin{aligned}\dot{x}(t) &= 4\delta(t+1)u(1-t) + 4u(t+1)\delta(1-t)\\ &= 4\delta(t+1) - 4\delta(1-t)\\ &= 4\delta(t+1) - 4\delta(t-1)\end{aligned}$$

(The first step makes use of the product rule for differentiation, the second step uses the signal-multiplication rule for impulses, and the last step uses the evenness of the impulse.)

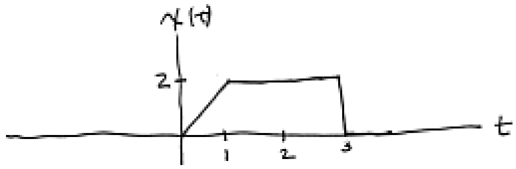
Of course, regardless of the approach taken, graphical methods easily verify

$$\int_{-\infty}^{t} \dot{x}(\tau)\, d\tau = x(t)$$

in this example. Note that the constant of integration is taken to be zero since it is known that the signal $x(t)$ is zero for $t < -1$.

Example The signal shown below can be written as

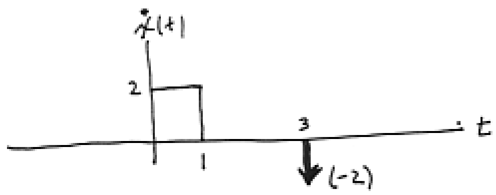
$$x(t) = 2r(t) - 2r(t-1) - 2u(t-3)$$



Again, the derivative is straightforward to compute,

$$\dot{x}(t) = 2u(t) - 2u(t-1) - 2\delta(t-3)$$

and sketch,



Graphical interpretations of differentiation support these computations.

While representation in terms of linear combinations of singularity signals leads to convenient shortcuts for some purposes, caution should be exercised. In the examples so far, well-behaved energy signals have been represented as linear combinations of signals that are power signals, singularity signals, and unbounded signals. This can introduce complications in some contexts.

Sometimes we use these generalized calculus ideas for signals are nonzero for infinite time intervals.

Example A right-sided cosine signal can be written as

$$x(t) = \cos(2t)\, u(t)$$

Then differentiation using the product rule, followed by the impulse multiplication rule, gives

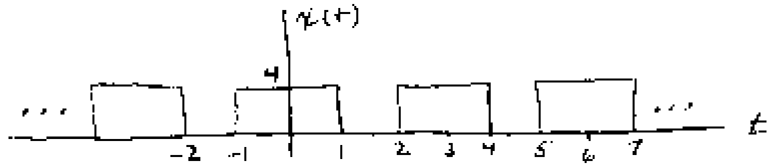
$$\begin{aligned}\dot{x}(t) &= -2\sin(2t)\, u(t) + \cos(2t)\, \delta(t) \\ &= -2\sin(2t)\, u(t) + \delta(t)\end{aligned}$$

You should graphically check that this is a consistent result, and that the impulse in $\dot{x}(t)$ is crucial in verifying the relationship

$$\int_{-\infty}^{t} \dot{x}(\tau)\, d\tau = x(t)$$

Example The periodic signal shown below can be written as

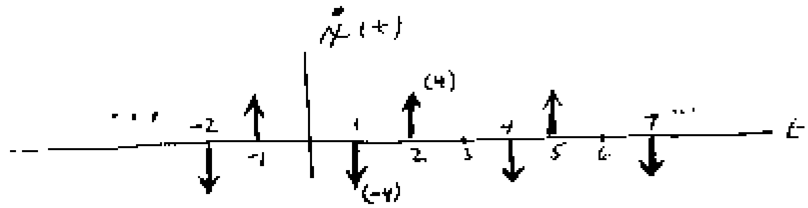
$$x(t) = \sum_{k=-\infty}^{\infty} \left[4u(t+1-3k) - 4u(t-1-3k)\right]$$



Then

$$\dot{x}(t) = \sum_{k=-\infty}^{\infty} \left[4\delta(t+1-3k) - 4\delta(t-1-3k)\right]$$

by generalized differentiation, and a sketch of $\dot{x}(t)$ is shown below.



Exercises

1. Determine whether the following signals are periodic, and if so determine the fundamental period.

(a) $x(t) = e^{-(\pi - 2j)t}$

(b) $x(t) = e^{-\pi + 2jt}$

(c) $x(t) = 3e^{j4t} - 2e^{j5t}$

(d) $x(t) = e^{j\frac{7}{2}t} + e^{j7t}$

(e) $x(t) = 6e^{-j2t} + 4e^{j7(t-1)} - 3e^{j6(t+1)}$

(f) $x(t) = \sum_{k=-\infty}^{\infty} \left[e^{-(t-2k)}u(t-2k) - e^{-(t-2k)}u(t-2k-1)\right]$

2. Simplify the following expressions and sketch the signal.

(a) $x(t) = \delta(t-2)r(t) + \frac{d}{dt}[u(t) - r(t-1)]$

(b) $x(t) = \delta(t)\delta(t-1) - e^{t+3}\delta(t+2) + u(1-t)r(t)$

(c) $x(t) = \int_{-\infty}^{t} u(\tau - 3)\, d\tau - 2r(t-4) + r(t-5)$

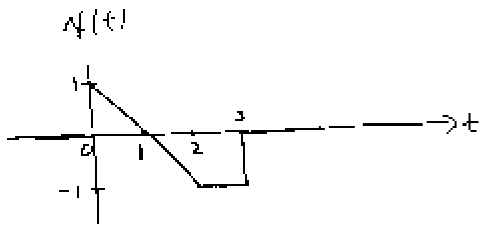
(d) $x(t) = \int_{-\infty}^{t} \delta(\tau + 2)\, d\tau + \frac{d}{dt}\left[u(t+2)r(t)\right]$

3. Sketch the signal $x(t)$ and compute and sketch $\int_{-\infty}^{t} x(\sigma)\, d\sigma$. Check your integration by "graphical differentiation."

(a) $x(t) = u(t) - u(t-1) + 2\delta(t-2) - 3\delta(t-3) + \dot{\delta}(t-4)$
(b) $x(t) = 3u(t) - 3u(t-4) - 12\delta(t-5)$
(c) $x(t) = -2u(t+1) + 3u(t) - u(t-2)$
(d) $x(t) = u(t) + \delta(t-1) - 2\delta(t-2) + \delta(t-4)$

4. Sketch the signal $x(t)$ and compute and sketch $\dot{x}(t)$. Check your derivative by "graphical integration."
(a) $x(t) = r(t+1) - r(t) + u(t) - 3u(t-1) + u(t-2)$
(b) $x(t) = 2r(t+1) - 4r(t) + 2r(t-1)$
(c) $x(t) = 2u(t-1) - (2/3)r(t-1) + (2/3)r(t-4)$

5. Write a mathematical expression for the signal $x(t)$ shown below.



Compute and sketch $\dot{x}(t)$, the generalized derivative of $x(t)$.

Notes for Signals and Systems

3. 1 The Class of DT Exponential Signals

Consider a discrete-time signal

$$x[n] = c\,e^{an}$$

where both c and a are complex numbers, and, as usual, the integer index, or sample number, covers the range $-\infty < n < \infty$. To more conveniently interpret $x[n]$, write c in polar form, and a in rectangular form,

$$c = |c|e^{j\phi_o} \quad , \quad a = \sigma_o + j\omega_o$$

where $\phi_o = \angle c$ and we have chosen customary notations for the rectangular form of a. Then

$$x[n] = |c|e^{j\phi_o}\; e^{(\sigma_o + j\omega_o)n}$$
$$= |c|e^{\sigma_o n}\; e^{j(\omega_o n + \phi_o)}$$

Using Euler's formula, we can write this signal in rectangular form as

$$x[n] = |c|e^{\sigma_o n}\cos(\omega_o n + \phi_o) + j\,|c|e^{\sigma_o n}\sin(\omega_o n + \phi_o)$$

This expression is similar to the continuous-time case, but differences appear upon closer inspection. We need only consider three cases in detail.

Special Case 1: Suppose both c and a are real. That is, $\omega_o = 0$ and ϕ_o has value either 0 or π. Then,

$$x[n] = c\,e^{\sigma_o n}$$

and we have the familiar interpretation of an exponentially-decreasing (if $\sigma_o < 0$) or exponentially-increasing $\sigma_o > 0$ sequence of values, with the obvious consideration of the sign of c.

Special Case 2: Suppose c is real and a has the special form

$$a = \sigma_o + j\,(2k+1)\pi$$

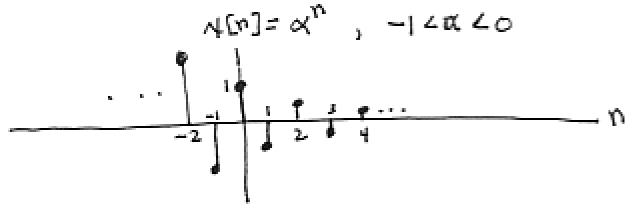
where k is an integer. In this case,

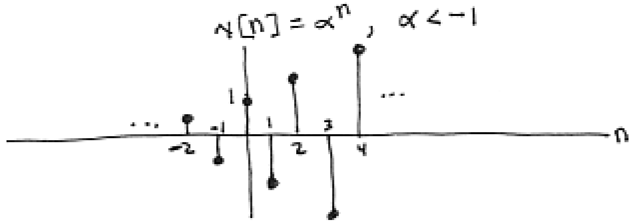
$$x[n] = c\,e^{[\sigma_o + j(2k+1)\pi]n} = c\,e^{\sigma_o n}\; e^{j(2k+1)\pi}$$
$$= c\,e^{\sigma_o n}\,(-1)^n = c\,(-e^{\sigma_o})^n$$

Or, more simply, we can write

$$x[n] = c\,\alpha^n$$

where $\alpha = -e^{\sigma_o}$ is a real negative number. The appearance of $x[n]$ is a bit different in this case, as shown below for $c = 1$.





Of course, these first two special cases can be combined by considering that α might be either positive or negative.

Special Case 3: Suppose *c* is complex and *a* is purely imaginary. That is, $\sigma_o = 0$, and we are considering a discrete-time *phasor*. Then

$$\begin{aligned} x[n] &= |c| e^{j\phi_o} e^{j\omega_o n} \\ &= |c| e^{j(\omega_o n + \phi_o)} \\ &= |c| \cos(\omega_o n + \phi_o) + j|c| \sin(\omega_o n + \phi_o) \end{aligned}$$

Since the independent variable, *n*, is viewed as a sample index, units of ω_o typically are *radians/sample* for discrete-time signals. In order to simplify further discussion, we assume that *c = 1*.

The first difference from the continuous-time case involves the issue of periodicity. The signal

$$x[n] = e^{j\omega_o n}$$

is periodic if and only if there is a positive integer *N* such that

$$e^{j\omega_o (n+N)} = e^{j\omega_o n}$$

for all integer *n*. This will hold if and only if *N* satisfies

$$e^{j\omega_o N} = 1$$

that is, if and only if $\omega_o N = m2\pi$ for some integer *m*. In words, *x*[*n*] is periodic if and only if the frequency ω_o is a rational multiple of 2π,

$$\omega_o = \frac{m}{N} 2\pi$$

for some integers *m* and *N*. If this is satisfied, then the fundamental period of the signal is obtained when the smallest value of *N* is found for which this expression holds. Obviously, this occurs when *m*, *N* are relatively prime, that is, *m* and *N* have no common integer factors other than unity.

Example $x[n] = e^{j2n}$ is not periodic, since $\omega_o = 2$ cannot be written as a rational multiple of 2π.

Example $x[n] = e^{j\frac{\pi}{8}n}$ is periodic, with fundamental period 16, since

$$\frac{\pi}{8} = \frac{1}{16} 2\pi$$

The second major difference between continuous and discrete-time complex exponentials concerns the issue of frequency. In continuous time, as the frequency ω_o increases, the phasor rotates more rapidly. But in discrete time, consider what happens when the frequency is raised or lowered by 2π:

$$e^{j(\omega_o \pm 2\pi)n} = e^{j\omega_o n} e^{\pm j2\pi n} = e^{j\omega_o n}$$

That is, the complex exponential signal doesn't change. Clearly, raising or lowering the frequency by any integer multiple of 2π has the same (absence of) effect. Thus we need only consider discrete-time frequencies in a single 2π range. Most often the range of choice is $-\pi < \omega_o \le \pi$ for reasons of symmetry, though sometimes the range
$0 \le \omega_o < 2\pi$ is convenient.

As a final observation, in continuous time a phasor rotates counterclockwise if the frequency ω_o is positive, and clockwise if the frequency is negative. However there is no clear visual interpretation of direction of rotation depending on the sign of frequency in the discrete time case.

To balance these complications in comparison with the continuous-time case, the issue of periodicity of sums of periodic discrete-time phasors is relatively simple.

Theorem For the complex-valued signal

$$x[n] = c_1 e^{j\omega_1 n} + c_2 e^{j\omega_2 n}$$

suppose both frequencies are rational multiples of 2π,

$$\omega_1 = (m_1 / N_1)2\pi \,, \quad \omega_2 = (m_2 / N_2)2\pi$$

Then $x[n]$ is periodic with period (not necessarily fundamental period) given by $N = N_1 N_2$.

Proof By direct calculation, using the claimed period N,

$$\begin{aligned}
x[n+N] &= c_1 e^{j\omega_1 (n+N)} + c_2 e^{j\omega_2 (n+N)} \\
&= c_1 e^{j\omega_1 n}\, e^{j\frac{m_1}{N_1}2\pi N_1 N_2} + c_2 e^{j\omega_2 n}\, e^{j\frac{m_2}{N_2}2\pi N_1 N_2} \\
&= c_1 e^{j\omega_1 n}\, e^{jm_1 N_2 2\pi} + c_2 e^{j\omega_2 n}\, e^{jm_2 N_1 2\pi} \\
&= c_1 e^{j\omega_1 n} + c_2 e^{j\omega_2 n} \\
&= x[n]
\end{aligned}$$

for any n.

The applet below illustrates the behavior of discrete-time phasors for various choices of frequency, ω_o, and suggests a number of exercises.

Discrete-Time Frequency

Remark The various analyses of phasor properties can also be carried out for real trigonometric signals, though the reasoning often requires trig identities and is more involved than for phasors. For example, suppose

$$x[n] = \cos(\omega_o n)$$

is periodic with period *N*. That is, for all integers *n*,

$$\cos[\omega_o(n+N)] = \cos(\omega_o n)$$

This can be written as

$$\cos(\omega_o n + \omega_o N) = \cos(\omega_o n)\cos(\omega_o N) - \sin(\omega_o n)\sin(\omega_o N)$$
$$= \cos(\omega_o n)\,, \quad -\infty < n < \infty$$

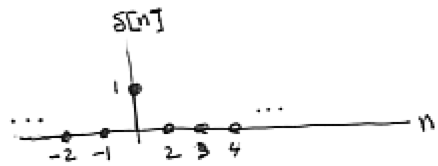
We conclude from this that *N* must be such that $\cos(\omega_o N) = 1,\ \sin(\omega_o N) = 0$. Thus $\omega_o N = m2\pi$ for some integer *m*, that is, ω_o must be a rational multiple of 2π.

3.2 The Class of DT Singularity Signals

The basic discrete-time singularity signal is the *unit pulse*,

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & otherwise \end{cases}$$

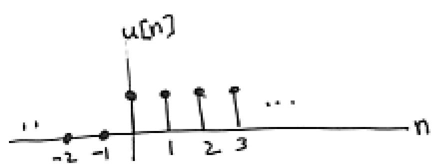
Contrary to the continuous-time case, there is nothing "generalized" about this simple signal. Graphically, of course, $\delta[n]$ is a lonely lollypop at the origin:



The discrete-time unit-step function is

$$u[n] = \begin{cases} 1, & n \ge 0 \\ 0, & n < 0 \end{cases}$$

and again there are no technical issues here. In particular, the value of $u[0]$ is unity, unlike the continuous-time case where we decline to assign an immoveable value to $u(0)$. Graphically, we have



The unit step can be viewed as the running sum of the unit pulse,

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

Changing summation variable from *k* to $l = n - k$ gives an alternate expression

$$u[n] = \sum_{l=0}^{\infty} \delta[n-l]$$

This process can be continued to define the *unit ramp* as the running sum of the unit step, though there is a little adjustment involved in the upper limit of the sum, since $u[0]=1$ and $r[0]=0$:

$$r[n]=\sum_{k=-\infty}^{n-1} u[k]=\begin{cases} n, & n\ge 0 \\ 0, & n<0 \end{cases}$$

We will have no reason to pursue further iterations of running sums. One reason is that simple discrete-time signals can be written as sums of amplitude scaled and time shifted unit pulses, and there is little need to write signals in terms of steps, ramps, and so on.

Discrete-time singularity signals also have sifting and multiplication properties similar to the continuous-time case, though no "generalized" interpretations are needed. It is straightforward to verify that

$$\sum_{n=-\infty}^{\infty} x[n]\delta[n-n_o]=x[n_o]$$

which is analogous to the continuous-time sift

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_o)\,dt=x(t_o)$$

Also,

$$x[n]\delta[n-n_o]=x[n_o]\delta[n-n_o]$$

is a discrete-time version of the multiplication rule

$$x(t)\delta(t-t_o)=x(t_o)\delta(t-t_o)$$

However, unlike the continuous-time case, the time-scaled unit pulse, $\delta[an]$, where *a* is a nonzero integer, is identical to $\delta[n]$, as is easily verified.

Exercises

1. Determine if the following signals are periodic, and if so compute the fundamental period.

(a) $x[n]=e^{j\frac{20\pi}{3}n}$

(b) $x[n]=e^{j4\pi n}-e^{-j\frac{\pi}{4}n}$

(c) $x[n]=3e^{j\frac{7}{3}n}$

(d) $x[n]=1+e^{j\frac{4}{5}\pi n}$

(e) $x[n]=e^{j\frac{5}{7}\pi n}+e^{-j\frac{3}{4}\pi n}$

2. Consider the signal $x[n]=c_1e^{j\omega_1 n}+c_2e^{j\omega_2 n}$ where both frequencies are rational multiples of 2π,

$$\omega_1=(m_1/N_1)2\pi\,,\qquad \omega_2=(m_2/N_2)2\pi$$

Suppose that N is a positive integer such that

$$N=k_1N_1=k_2N_2$$

for some integers k_1, k_2. Show that $x[n]$ has period N. (Typically $N<N_1N_2$, as used in the theorem in Section 3.1.)

3. Simplify the following expressions and sketch the signal.

(a) $x[n] = \sum_{k=-\infty}^{n} 3\delta[k-2] + \delta[n+1]\cos(\pi n)$

(b) $x[n] = \sum_{k=-\infty}^{\infty} 4\delta[n-k]\, e^{3k}$

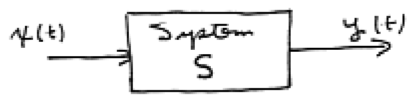
(c) $x[n] = r[n-3]\delta[n-5] + \sum_{k=-\infty}^{n} 3\delta[n-k]$

(d) $x[n] = \cos(\pi n)\left[\delta[n] - \delta[n-1]\right] - \delta^{3}[n] + \sum_{k=-\infty}^{n} u[k-3]$

Notes for Signals and Systems

4.1 Introduction to Systems

A continuous-time *system* produces a new continuous-time signal (the *output* signal) from a provided continuous-time signal (the *input* signal). Attempting a formal definition of a system is a tedious exercise in avoiding circularity, so we will abandon precise articulation and rely on the intuition that develops from examples. We represent a system “S” with input signal $x(t)$ and output signal $y(t)$ by a box labeled as shown below.



Since a system maps signals into signals, the output signal at any time *t* can depend on the input signal values at all times. We use the mathematical notation

$$y(t) = S(x)(t)$$

to emphasize this fact.

Remark There are many notations for a system in the literature. For example, some try to emphasize that a system maps signals into signals by using square brackets,

$$y(t) = S[x(t)]$$

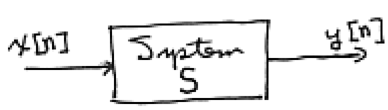
Perhaps this continues to tempt the interpretation that, for example, $y(0) = S[x(0)]$, i.e., $y(0)$ depends only on $x(0)$. But, the notation is designed to emphasize that for the input signal “*x*” the output signal is “*S*(*x*),” and the value of this output signal at, say $t = 0$, is $y(0) = S(x)(0)$.

Example The running integral is an example of a system. A physical interpretation is a capacitor with input signal the current $x(t)$ through the capacitor, and output signal the voltage $y(t)$ across the capacitor. Then we have, assuming unit capacitance,

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

In this case, the output at any time t_1 depends, on input values for all $t \le t_1$. Specifically, at any time t_1, $y(t_1)$ is the accumulated net area under $x(t)$ for $-\infty < t \le t_1$.

In the discrete-time case, we use an entirely similar notation, writing $y[n] = S(x)[n]$ and representing a system by the diagram shown below.



4.2 System Properties

The discussion of properties of systems will be a bit tentative at this point, in part because the notion of a system is so general that it is difficult to include all the details, and in part because the mathematical description of a system might presume certain properties of allowable input signals.

For example, the input signal to a running-integrator system must be sufficiently well behaved that the integral is defined. We can be considerably more precise when we consider specific classes of systems that admit particular types of mathematical descriptions. In the interim the intent is mainly to establish some intuition concerning properties of systems in general.

We proceed with a list of properties, phrased in the continuous-time case.

- *Causal System* A system is *causal* if the output signal value at any time t depends only on input signal values for times no larger than t. Examples of causal systems are

$$y(t) = 3x(t-2), \quad y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau, \quad y(t) = x^3(t)$$

Examples of systems that are not causal are

$$y(t) = x(2), \quad y(t) = 3x(t+2), \quad y(t) = \int_{-\infty}^{t+1} x(\tau)\, d\tau$$

- *Memoryless System* A system is *memoryless* if the output value at any time t depends only on the input signal value at that same time, t. A memoryless system is always causal, though the reverse is, of course, untrue. Examples of memoryless systems are

$$y(t) = 2x(t), \quad y(t) = x^2(t), \quad y(t) = te^{x(t)}$$

- *Time-Invariant System* A system is *time invariant* if for every input signal $x(t)$ and corresponding output signal $y(t)$ the following property holds. Given any constant, t_o, the input signal $\tilde{x}(t) = x(t-t_o)$ yields the output signal $\tilde{y}(t) = y(t-t_o)$. This is sometimes called "shift invariance," since any time shift of an input signal results in the exact same shift of the output signal. Examples of time-invariant systems are

$$y(t) = \sin(x(t)), \quad y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau, \quad y(t) = 3x(t-2)$$

Examples of systems that are not time invariant are

$$y(t) = \sin(t)\, x(t), \quad y(t) = \int_{-\infty}^{t} \tau\, x(\tau)\, d\tau$$

To check if a system is time invariant requires application of the defining condition. For example, for

$$y(t) = \int_{-\infty}^{t} \tau x(\tau)\, d\tau$$

we consider the input signal $\tilde{x}(t) = x(t-t_o)$, where t_o is any constant. The corresponding response computation begins with

$$\tilde{y}(t) = \int_{-\infty}^{t} \tau\, \tilde{x}(\tau)\, d\tau = \int_{-\infty}^{t} \tau\, x(\tau - t_o)\, d\tau$$

To compare this to $y(t-t_o)$, it is convenient to change the variable of integration to $\sigma = \tau - t_o$. This gives

$$\tilde{y}(t) = \int_{-\infty}^{t-t_o} (\sigma + t_o)\, x(\sigma)\, d\sigma$$

which is not the same as

$$y(t - t_o) = \int_{-\infty}^{t-t_o} \tau x(\tau)\, d\tau$$

Therefore the system is not time invariant.

- *Linear System* A system is *linear* if for every pair of input signals $x_1(t), x_2(t)$,

with corresponding output signals $y_1(t), y_2(t)$, the following holds. For every constant *b*, the response to the input signal $x(t) = bx_1(t) + x_2(t)$ is $y(t) = by_1(t) + y_2(t)$. (This is more concise than popular two-part definitions of linearity in the literature. Taking $b = 1$ yields the *additivity* requirement that the response to $x(t) = x_1(t) + x_2(t)$ be $y(t) = y_1(t) + y_2(t)$. And taking $x_2(t) = x_1(t)$ gives the *homogeniety* requirement that the response to $x(t) = (b+1)x_1(t)$ should be $y(t) = (b+1)y_1(t)$ for any constant *b*. Examples of linear systems are

$$y(t) = e^t x(t), \quad y(t) = 3x(t-2), \quad y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

Examples of systems that are "nonlinear" are

$$y(t) = \int_{-\infty}^{t} x^2(\sigma)\, d\sigma, \quad y(t) = 1 + x(t), \quad y(t) = \cos(x(t))$$

Remark It should be noted that for a linear system the response to the zero input is the zero output signal. To see this, simply take $x_1(t) = x_2(t)$ (so that $y_1(t) = y_2(t)$) and $b = -1$ in the definition of linearity.

- *Stable System* Recalling the definition of a bounded signal in Section1.6, a

system is *stable* (or *bounded-input, bounded-output stable*) if every bounded input signal yields a bounded output signal. In detail, for any input signal $x(t)$ such that $|x(t)| < M$ for all *t*, where M is a constant, there is aother constant *P* such that the corresponding output signal satisfies $|y(t)| < P$ for all *t*. Examples of stable systems are

$$y(t) = e^{x(t)}, \quad y(t) = \frac{x(t-2)}{t^2+1}, \quad y(t) = \sin(t)\, x(t)$$

Examples of "unstable" systems are

$$y(t) = e^t x(t), \quad y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

- *Invertible System* A system is *invertible* if the input signal can be uniquely

determined from knowledge of the output signal. Examples of invertible systems are

$$y(t) = x^3(t), \quad y(t) = 3x(t-2) + 4t$$

The thoughtful reader will be justifiably nervous about this definition. Invertibility of a mathematical operation requires two features: the operation must be *one-to-one* and also *onto*. Since we have not established a class of input signals that we consider for systems, or a corresponding class of output signals, the issue of "onto" is left vague. And since we have decided to ignore or reassign values of a signal at isolated points in time for reasons of simplicity or convenience, even the issue of “one-to-one” is unsettled.

Determining invertibility of a given system can be quite difficult. Perhaps the easiest situation is showing that a system is *not* invertible by exhibiting two legitimately different input signals that yield the same output signal. For example, $y(t) = x^2(t)$ is not invertible because constant input signals of $x(t) = 1$ and $x(t) = -1$, for all *t*, yield identical output signals. As another example, the system

$$y(t) = \frac{d}{dt} x(t)$$

is not invertible since $\tilde{x}(t) = 1 + x(t)$ yields the same output signal as *x*(*t*). As a final example, in a benign setting the running-integrator system

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

is invertible by the fundamental theorem of calculus:

$$\frac{d}{dt} \int_{-\infty}^{t} x(\tau)\, d\tau = x(t)$$

But the fact remains that technicalities are required for this conclusion. If two input signals differ only at isolated points in time, the output signals will be identical, and thus the system is not invertible if we consider such input signals to be legitimately different.

All of these properties translate easily to discrete-time systems. Little more is required than to replace parentheses by square brackets and *t* by *n*. But regardless of the time domain, it is important to note that these are *input-output properties* of systems. In particular, nothing is being stated about the internal workings of the system, everything is stated in terms of input signals and corresponding output signals.

Finally it is worthwhile to think of how you would ascertain whether a given physical system, for which you do not have a mathematical description, has each of the properties we consider. That is, what input signals would you apply, what measurements of the response would you take, and what use you would make of these measurements.

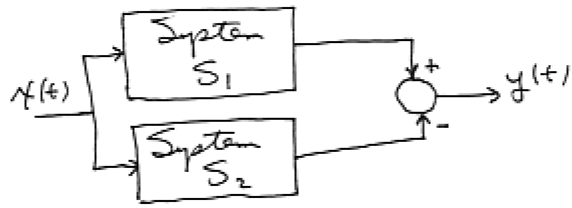
4.3 Interconnections of Systems – Block Diagrams

Engineers often connect systems, sometimes called subsystems in this context, together to form new systems. An immediate question is how to mathematically describe the input-output behavior of the overall system in terms of the subsystem descriptions. Of course these connections correspond to mathematical operations on the functions describing the subsystems, and there are a few main cases to consider.

We begin with the two subsystems shown below as “blocks” ready for connection,

and consider the following types of connections.

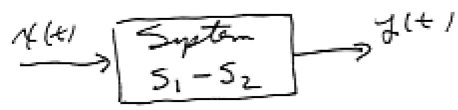
- *Additive parallel connection*



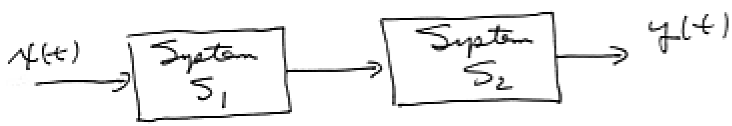
In this *block diagram*, as a particular example, the circular element shown represents the signed addition of the output signals of the two subsystems. This connection describes a new system with output given by

$$y(t) = S_1(x)(t) - S_2(x)(t) = (S_1 - S_2)(x)(t)$$

Thus the overall system is represented by the function $(S_1 - S_2)(x) = S_1(x) - S_2(x)$, and we can represent it as a single block:



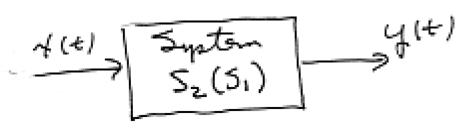
- *Series* or *cascade connection*



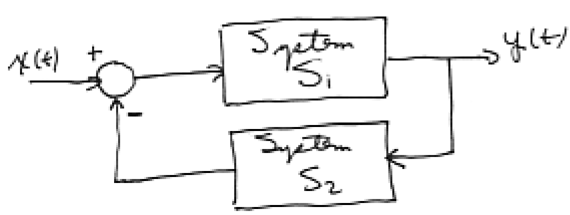
The mathematical description of the overall system corresponds to the composition of the functions describing the subsystems, according to

$$y(t) = S_2(S_1(x))(t)$$

Thus the overall system can be represented as the block diagram



- *Feedback connection*



The feedback connection is more complicated than the others, but extremely important. To attempt development of a mathematical description of the system described by this connection, we begin by noting that the output signal is the response of the system S_1 with input signal that is the output of the summer. That is, input signal to S_1 is $x(t) - S_2(y)(t)$. Therefore,

$$y(t) = S_1(x - S_2(y))(t)$$

and we see that the feedback connection yields an equation involving both the input and output signals. Unfortunately, without further mathematical assumptions on the two functions S_1 and S_2, we cannot solve this equation for $y(t)$ in terms of $x(t)$ to obtain a mathematical description of the overall system of the form

$$y(t) = S(x)(t)$$

This indicates the sophisticated and subtle nature of the feedback connection of systems, and we return to this issue in the sequel.

Remark In our discussion of interconnections of systems, an underlying assumption is that the behavior of the various subsystems is not altered by the connection to other subsystems. In electrical engineering, this is often expressed as "there are no loading effects." This assumption is not always satisfied in practice. A second assumption is that domain and range conditions are met. For example, in the cascade connection, the output of S_1 must satisfy any assumption required on the input to S_2. If the system S_2 takes the square root of the input signal, $S_2(x)(t) = \sqrt{x(t)}$, then the output of S_1 must never become negative.

The basic interconnections of discrete-time systems are completely similar to the continuous-time case.

Exercises

1. Determine if each of the following systems is causal, memoryless, time invariant, linear, or stable. Justify your answer.

(a) $y(t) = 2x(t) + 3x^2(t-1)$

(b) $y(t) = \cos^2(t)\, x(t)$

(c) $y(t) = 2 + \int_{-\infty}^{t-1} x(t-\tau)\,d\tau$

(d) $y(t) = e^{-t} \int_{-\infty}^{t} e^{\tau} x(\tau)\,d\tau$

(e) $y(t) = \int_{-\infty}^{t} x(2\tau)\,d\tau$

(f) $y(t) = x(-t)$

(g) $y(t) = x(3t)$

(h) $y(t) = \int_{-\infty}^{t} e^{(t-\sigma)} x^2(\sigma)\,d\sigma$

(i) $y(t) = x(t) \int_{-\infty}^{t} e^{-\tau} x(\tau)\, d\tau$

(j) $y(t) = 3x(t+1) - 4$

(k) $y(t) = -3x(t) + \int_{0}^{t} 3x(\sigma) d\sigma$

(l) $y(t) = 3x(t) - |x(t-3)|$

2. Determine if each of the following systems is causal, memoryless, time invariant, linear, or stable. Justify your answer.

(a) $y[n] = 3x[n]x[n-1]$

(b) $y[n] = \sum_{k=n-2}^{n+2} x[k]$

(c) $y[n] = 4\,x[3n-2]$

(d) $y[n] = \sum_{k=-\infty}^{n} e^{k} u[k]\, x[n-k]$

(e) $y[n] = \sum_{k=n-3}^{n} \cos(x[k])$

3. Determine if each of the following systems is invertible. If not, specify two different input signals that yield the same output. If so, give an expression for the inverse system.

(a) $y(t) = \cos[x(t)]$

(b) $y(t) = x^{3}(t-4)$

4. Determine if each of the following systems is invertible. If not, specify two different input signals that yield the same output. If so, give an expression for the inverse system.

(a) $y[n] = \sum_{k=-\infty}^{n} x[k]$

(b) $y[n] = (n-1)x[n]$

(c) $y[n] = x[n] - x[n-1]$

5. For each pair of systems S_1, S_2 specified below, give a mathematical description of the cascade connection $S_2(S_1)$.

(a) $y_1[n] = x_1^2[n-2], \quad y_2[n] = 3x_2[n+2]$

(b) $y_1[n] = \sum_{k=-\infty}^{n} \delta[k] x_1[n-k], \quad y_2[n] = \sum_{k=-\infty}^{n} 2\delta[n-k] x_2[k]$

Notes for Signals and Systems

5.1 DT LTI Systems and Convolution

Discrete-time systems that are linear and time invariant often are referred to as LTI systems. LTI systems comprise a very important class of systems, and they can be described by a standard mathematical formalism. To each LTI system there corresponds a signal $h[n]$ such that the input-output behavior of the system is described by

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

This expression is called the *convolution sum* representation for LTI systems. In addition, the sifting property easily shows that $h[n]$ is the response of the system to a unit-pulse input signal. That is, for $x[n] = \delta[n]$,

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} \delta[k]h[n-k] = h[n]$$

Thus the input-output behavior of a discrete-time, linear, time-invariant system is completely described by the *unit-pulse response* of the system. If $h[n]$ is known, then the response to any input can be computed from the convolution sum.

Derivation It is straightforward to show that a system described by the convolution sum, with specified $h[n]$, is a linear and time-invariant system. Linearity is clear, and to show time invariance, consider a shifted input signal $\hat{x}[n] = x[n-n_o]$. The system response to this input signal is given by

$$\begin{aligned}\hat{y}[n] &= \sum_{k=-\infty}^{\infty} \hat{x}[k]\, h[n-k] \\ &= \sum_{k=-\infty}^{\infty} x[k-n_o]\, h[n-k]\end{aligned}$$

To rewrite this expression, change the summation index from k to $l = k - N$, to obtain

$$\begin{aligned}\hat{y}[n] &= \sum_{l=-\infty}^{\infty} x[l]\, h[n-n_o-l] \\ &= y[n-n_o]\end{aligned}$$

This establishes time invariance.

It is less straightforward to show that essentially any LTI system can be represented by the convolution sum. But the convolution representation for linear, time-invariant systems can be developed by adopting a particular representation for the input signal and then enforcing the properties of linearity and time invariance on the corresponding response. The details are as follows.

Often we will represent a given signal as a linear combination of “basis” signals that have certain desirable properties for the purpose at hand. To develop a representation for discrete-time LTI systems, it is convenient to represent the input signal as a linear combination of shifted unit pulse signals: $\delta[n], \delta[n-1], \delta[n+1], \delta[n-2], \ldots$. Indeed it is easy to verify the expression

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Here the coefficient of the signal $\delta[n-k]$ in the linear combination is the value $x[k]$. Thus, for example, if $n = 3$, then the right side is evaluated by the sifting property to verify

$$\sum_{k=-\infty}^{\infty} x[k]\delta[3-k] = x[3]$$

We can use this signal representation to derive an LTI system representation as follows. The response of an LTI system to a unit pulse input, $x[n] = \delta[n]$, is given the special notation $y[n] = h[n]$. Then by time invariance, the response to a k–shifted unit pulse, $\tilde{x}[n] = \delta[n-k]$ is $\tilde{y}[n] = h[n-k]$. Furthermore, by linearity, the response to a linear combination of shifted unit pulses is the linear combination of the responses to the shifted unit pulses. That is, the response to $x[n]$, as written above, is

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

Thus we have arrived at the *convolution sum* representation for LTI systems. The convolution representation follows directly from linearity and time invariance – no other properties of the system are assumed (though there are some convergence issues that we have ignored). An alternate expression for the convolution sum is obtained by changing the summation variable from k to $l = n - k$:

$$y[n] = \sum_{k=-\infty}^{\infty} h[l]x[n-l]$$

It is clear from the convolution representation that if we know the unit-pulse response of an LTI system, then we can compute the response to any other input signal by evaluating the convolution sum. Indeed, we specifically label LTI systems with the unit-pulse response in drawing block diagrams, as shown below

The demonstration below can help with visualizing and understanding the convolution representation.

[Joy of Convolution (Discrete Time)]

Response Computation

Evaluation of the convolution expression, given $x[n]$ and $h[n]$, is not as simple as might be expected because it is actually a collection of summations, over the index k, that can take different forms for different values of n. There are several strategies that can be used for evaluation, and the main ones are reviewed below.

- *Analytic evaluation* When $x[n]$ and $h[n]$ have simple, neat analytical expressions,

and the character of the summation doesn't change in complicated ways as n changes, sometimes $y[n]$ can be computed analytically.

Example Suppose the unit pulse response of an LTI system is a unit ramp,

$$h[n] = r[n] = n\,u[n]$$

To compute the response of this system to a unit-step input, write the convolution representation as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} u[k](n-k)\,u[n-k]$$

$$= \sum_{k=0}^{\infty} (n-k)\,u[n-k]$$

Note that in the second line the unit-step $u[k]$ in the summand is removed, but the lower limit of the sum is raised to zero, and this is valid regardless of the value of n. Now, if $n < 0$, then the argument of the step function in the summand is negative for every $k \geq 0$. Therefore $y[n] = 0$ for $n < 0$. But, for $n \geq 0$, we can remove the step $u[n-k]$ from the summand if we lower the upper limit to n. Then

$$y[n] = \sum_{k=0}^{n} (n-k) = n + (n-1) + \cdots + 2 + 1 + 0$$

Using the very old trick of pairing the n with the 0, the $(n-1)$ with the 1, and so on, we see that each pair sums to n. Counting the number of pairs for even n and for odd n gives

$$y[n] = \begin{cases} \dfrac{n(n+1)}{2}, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

or, more compactly,

$$y[n] = \frac{n(n+1)}{2} u[n]$$

Remark Because of the prevalence of exponential signals, the following geometric-series formulas for discrete-time signal calculations are useful for analytic evaluation of convolution. For any complex number $\alpha \neq 1$,

$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$$

For any complex number satisfying $|\alpha| < 1$,

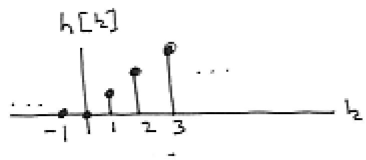
$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$$

- *Graphical method* This method is useful for more complicated cases. We simply plot the two signals, $x[k]$ and $h[n-k]$, in the summand versus k for the value of n of interest, then perform "lollypop-by-lollypop" multiplication and plot the summand, and then add up the lollypop values to obtain $y[n]$.

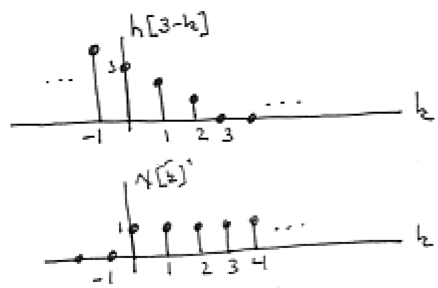
Example To rework the previous example by the graphical method, writing

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

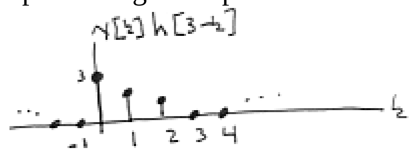
first plot $h[k]$ as shown,



Then, for the value of n of interest, flip and shift. For $n = 3$, we plot $h[3-k]$ and, to facilitate the multiplication, plot $x[k]$ immediately below:



Then lollypop-by-lollypop multiplication gives a plot of the summand,



Adding up the lollypop values gives

$$y[3] = 3 + 2 + 1 = 6$$

To compute, say, $y[4]$, slide the plot of $h[3-k]$ one sample to the right to obtain a plot of $h[4-k]$ and repeat the multiplication with $x[k]$ and addition. In simple cases such as this, there is little need to redraw because the pattern is clear. Even in complicated cases, it is often easy to identify ranges of n where $y[n] = 0$, because the plots of $x[k]$ and $h[n-k]$ are "non-overlapping." In the example, this clearly holds for $n < 0$.

- *LTI cleverness* The third method makes use of the properties of linearity and time invariance, and is well suited for the case where $x[n]$ has only a few nonzero values. Indeed, it is simply a specialization of the approach we took to the derivation of the convolution sum.

Example With an arbitrary $h[n]$, suppose that the input signal comprises three nonzero lollypops, and can be written as

$$x[n] = \delta[n] + 2\delta[n-1] - 3\delta[n-3]$$

Then linearity and time invariance dictate that

$$y[n] = h[n] + 2h[n-1] - 3h[n-3]$$

Depending on the form of the unit-pulse response, this can be evaluated analytically, or by graphical addition of plots of the unit-pulse response and its shifts and amplitude scales.

Remark The convolution operation can explode – fail to be well defined – for particular choices of input signal and unit-pulse response. For example, with $x[n] = h[n] = 1$, for all n, there is no value of n for which $y[n]$ is defined, because the convolution sum is infinite for every n. In our derivation, we did not worry about convergence of the summation. This again is a consequence of our decision not to be precise about classes of allowable signals, or, more mathematically, domains and ranges. The diligent student must always be on the look out for such anomalies. Furthermore, there are LTI systems that cannot be described by a convolution sum, though these are more in the nature of mathematical oddities than engineering legitimacies. In any case, this is the reason we say that "essentially" any LTI system can be described by the convolution sum.

5.2 Properties of Convolution – Interconnections of DT LTI Systems

The convolution of two signals yields a signal, and this obviously is a mathematical operation – a sort of "weird multiplication" of signals. This mathematical operation obeys certain algebraic properties, and these properties can be interpreted as properties of systems and their interconnections.

To simplify matters, we adopt a shorthand "star" notation for convolution and write

$$y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Note that since, for any n, the value of $y[n]$ in general depends on all values of the signals $x[n]$ and $h[n]$, we use the more general operator notation style, In particular, we do *not* write $y[n] = x[n] * h[n]$ because of the temptation to conclude that, for example, $y[2] = x[2] * h[2]$.

Algebraic properties of the "star operation" are discussed below.

- *Commutativity* Convolution is commutative. That is, $(x * h)[n] = (h * x)[n]$, or, in complete detail,

$$\sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]\,, \quad \textit{for all } n$$

The proof of this involves the first standard rule for proofs in this course: Use change of variable of summation. Beginning with the left side, replace the summation variable k by $q = n - k$. Then as $k \to \pm\infty$, $q \to \mp\infty$, but (unlike integration) it does not matter whether we sum from left-to-right or right-to-left. Thus

$$(x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{q=\infty}^{-\infty} x[n-q]h[q]$$

$$= \sum_{q=-\infty}^{\infty} h[q]x[n-q] = (h * x)[n]$$

Using this result, there are two different ways to describe in words the role of the unit-pulse response values in the input-output behavior of an LTI system. The value of $h[n-k]$ determines how the n^{th} value of the output signal depends on the k^{th} value of the input signal. Or, the value of $h[q]$ determines how the value of $y[n]$ depends on the value of $x[n-q]$.

- *Associativity* Convolution is associative. That is,

$$(x * (h_1 * h_2))[n] = ((x * h_1) * h_2)[n]$$

The proof of this property is a messy exercise in manipulating summations, and it is omitted.

- *Distributivity* Convolution is distributive (with respect to addition). That is,

$$(x*(h_1+h_2))[n]=(x*h_1)[n]+(x*h_2)[n]$$

Of course, distributivity is a restatement of part of the linearity property of LTI systems and so no proof is needed. The remaining part of the linearity condition is written in the new notation as follows. For any constant *b*,

$$((bx)*h)[n]=b\,(x*h)[n]$$

- *Shift Property* This is simply a restatement of the time-invariance property, though the notation makes it a bit awkward. For any integer n_o , if $\hat{x}[n]=x[n-n_o]$, then

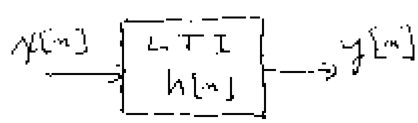
$$\left(\hat{x}*h\right)[n]=\left(x*h\right)[n-n_o]$$

- *Identity* It is worth noting that the "star" operation has the unit pulse as an identity element. Namely,

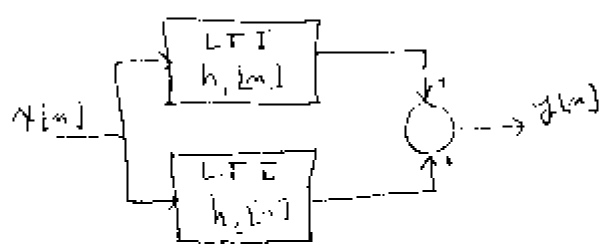
$$(x*\delta)[n]=x[n]$$

This can be interpreted in system-theoretic terms as the fact that the identity system, $y[n]=x[n]$ has the unit-pulse response $h[n]=\delta[n]$. Also we can write $\left(\delta*\delta\right)[n]=\delta[n]$, an expression that says nothing more than: The unit pulse is the unit-pulse response of the system whose unit-pulse response is a unit pulse.

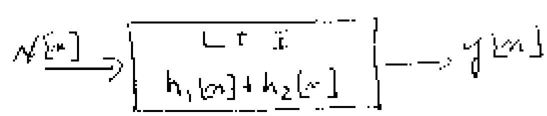
These algebraic properties of the mathematical operation of convolution lead directly to methods for describing the input-output behavior of interconnections of LTI systems. Of course we use block diagram representations to describe interconnections, but for LTI systems we label each block with the corresponding unit-pulse response. For example,



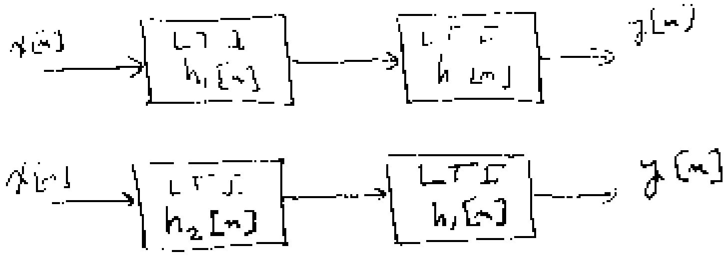
Distributivity implies that the interconnection below



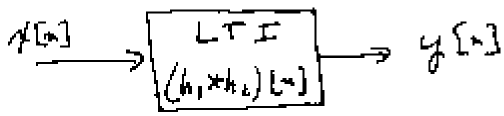
has the same input-output behavior as the system



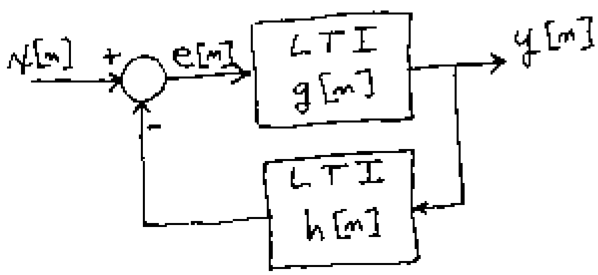
Commutativity and associativity imply that the interconnections



both have the same input-output behavior as the system



Finally, we analyze the feedback connection



as follows, in an attempt to obtain a description for its input-output behavior. With the intermediate signal $e[n]$ labeled as shown, we can write

$$y[n] = (g * e)[n]$$

and

$$e[n] = x[n] - (h * y)[n]$$

as descriptions of the interconnection. Substituting the second into the first gives

$$y[n] = (g * x)[n] - (g * h * y)[n]$$

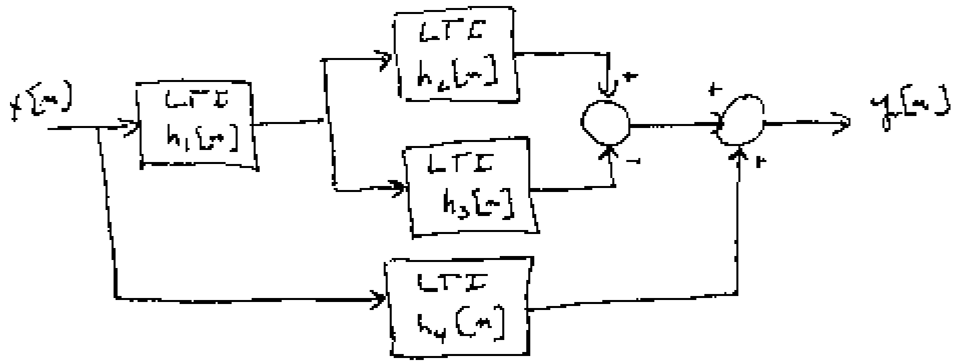
or, writing $y[n] = (\delta * y)[n]$ we get

$$\big((\delta - g * h) * y\big)[n] = (g * x)[n]$$

However, we cannot solve for $y[n]$ on the left side unless we know that the LTI system with unit-pulse response $(\delta - g * h)[n]$ is invertible. Let's stop here, and return to the problem of describing the feedback connection after developing more tools.

But for systems without feedback, the algebraic rules for the convolution operation provide an easy formalism for simplifying block diagrams. Typically it is easiest to start at the output signal and write descriptions of the intermediate signals (labeled if needed) while working back toward the input signal.

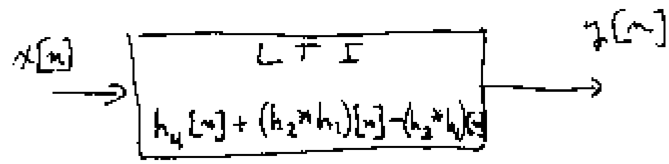
Example For the interconnected system shown below, there is no need to label internal signals as the structure is reasonably transparent.



The output signal can be written as

$$y[n] = (h_4 * x)[n] + (h_2 * h_1 * x)[n] - (h_3 * h_1 * x)[n]$$
$$= \big((h_4 + h_2 * h_1 - h_3 * h_1) * x\big)[n]$$

Thus the input-output behavior of the system is identical to the input-output behavior of the system



5.3 DT LTI System Properties

Since the input-output behavior of a discrete-time LTI system is completely characterized by its unit-pulse response, $h[n]$, via the convolution expression

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

the input-output properties of the system can be characterized very precisely in terms of properties of $h[n]$.

- *Causal System* An LTI system is causal if and only if $h[n] = 0$ for $n < 0$, that is,

if and only if $h[n]$ is right sided.

The proof of this is quite easy from the convolution expression. If the unit-pulse response is right sided, then the convolution expression simplifies to

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k]$$

and, at any value of n, the value of $y[n]$ depends only on the current and earlier values of the input signal. If the unit-pulse response is not right sided, then it is easy to see that the value of $y[n]$ at a particular n depends on future values of the input signal.

- *Memoryless System* An LTI system is memoryless if and only if $h[n] = c\delta[n]$, for some constant c. Again, a proof is quite easy to argue from the convolution expression.

- *Stable System* An LTI system is (bounded-input, bounded-output) stable if and only if the unit-pulse response is absolutely summable. That is,

$$\sum_{n=-\infty}^{\infty} |h[n]|$$

is finite.

To prove this, suppose $x[n]$ is a bounded input, that is, there is a constant M such that $|x[n]| \le M$ for all n. Then the absolute value of the output signal satisfies

$$|y[n]| = |\sum_{k=-\infty}^{\infty} h[k]x[n-k]| \le \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]|$$
$$\le M \sum_{k=-\infty}^{\infty} |h[k]|$$

Therefore, if the absolute summability condition holds, the output signal is bounded for any bounded input signal, and we have shown that the system is stable.

To prove that stability of the system implies absolute summability requires considerable cleverness. Consider the input $x[n]$ defined by

$$x[-n] = \begin{cases} 1, & h[n] \ge 0 \\ -1, & h[n] < 0 \end{cases}$$

Clearly $x[n]$ is a bounded input signal, and the corresponding response $y[n]$ at $n = 0$ is

$$y[0] = \sum_{k=-\infty}^{\infty} h[k]\, x[-k] = \sum_{k=-\infty}^{\infty} |h[k]|$$

Since the system is stable, $y[n]$ is bounded, and therefore $y[0]$ is bounded, and therefore the unit-pulse response is absolutely summable.

Example The system with unit-pulse response

$$h[n] = (0.5)^n u[n]$$

is a stable system since

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} (0.5)^n = \frac{1}{1-0.5} = 2$$

On the other hand, the system with unit-pulse response

$$h[n] = u[n-1]$$

is unstable.

- *Invertible System* There is no simple characterization of invertibility in terms of the the unit-pulse response. However, in particular examples it is sometimes possible to compute the unit-pulse response of an inverse system, $h_I[n]$, from the requirement

$$(h * h_I)[n] = \delta[n]$$

This condition expresses the natural requirement that a system in cascade with its inverse should be the identity system.

Example To compute an inverse of the running summer, that is, the LTI system with unit pulse response $h[n]=u[n]$, we must find $h_I[n]$ that satisfies

$$\sum_{k=-\infty}^{\infty} u[k]\,h_I[n-k]=\delta[n]$$

Simplifying the summation gives

$$\sum_{k=0}^{\infty} h_I[n-k]=\delta[n]$$

It is clear that we should take $h_I[n]=0$, for $n<0$. Using this result, for $n=0$ the requirement is

$$\sum_{k=0}^{\infty} h_I[-k] \;=h_I[0]=1$$

For $n=1$ the requirement is

$$\sum_{k=0}^{\infty} h_I[1-k]=h_I[1]+h_I[0]=0$$

which gives $h_I[1]=-1$. Continuing for further values of *n*, it is clear that the inverse-system requirement is satisfied by taking all remaining values of $h_I[n]$ to be zero. Thus the inverse system has the unit pulse response

$$h_I[n]=\delta[n]-\delta[n-1]$$

Of course, it is easy to see that in general the output of this inverse system is the *first difference* of the input signal.

5.4 Response to Singularity Signals

The response of a DT LTI system to the basic singularity signals is quite easy to compute. If the system is described by

$$y[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

then the unit pulse response is simply $y[n]=h[n]$. If the input signal is a unit step, $x[n]=u[n]$, then

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} u[k]h[n-k]=\sum_{k=0}^{\infty} h[n-k] \\ &= \sum_{l=-\infty}^{n} h[l] \end{aligned}$$

In words, the unit-step response is the running sum of the unit-pulse response. Of course, if the system is causal, that is, the unit-pulse response is right sided, then

$$y[n] = \begin{cases} \sum_{k=0}^{n} h[k], & n \ge 0 \\ 0, & n < 0 \end{cases}$$

$$= \left(\sum_{k=0}^{n} h[k] \right) u[n]$$

For input signals that have only a small number of nonzero values, the basic approach of "LTI cleverness" can be applied to evaluate the response. That is, if the input signal can be written as

$$x[n] = x[n_0]\delta[n-n_0] + x[n_1]\delta[n-n_1] + \cdots + x[n_m]\delta[n-n_m]$$

then the response is given by

$$y[n] = x[n_0]h[n-n_0] + x[n_1]y[n-n_1] + \cdots + x[n_m]y[n-n_m]$$

5.5 Response to Exponentials (Eigenfunction Properties)

For important classes of LTI systems, the responses to certain types of exponential input signals have particularly simple forms. These simple forms underlie many approaches to the analysis of LTI systems, and we consider several variants, each of which requires slightly different assumptions on the LTI system. For historical reasons in mathematics, an input signal $x[n]$ is called an *eigenfunction* of the system if the corresponding output signal is simply a constant multiple of the input signal. (We do permit the constant to be complex, when considering complex input signals.)

- *Real Eigenfunctions* The response of a causal, stable LTI system to a growing exponential input signal is a constant multiple of the exponential, where the constant depends on the exponent. To work this out in detail, suppose the LTI system

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

is causal and stable, that is, $h[n]$ is right-sided and absolutely summable. Furthermore, suppose the input signal is the real exponential signal

$$x[n] = e^{\sigma_o n}, \quad -\infty < n < \infty$$

where $\sigma_o \ge 0$. Then the response computation becomes

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, e^{\sigma_o (n-k)} = \left(\sum_{k=-\infty}^{\infty} h[k]\, e^{-\sigma_o k} \right) e^{\sigma_o n}$$

Using the causality and stability assumptions on the system, the assumption that $\sigma_o \ge 0$, and basic properties of absolute values,

$$\left|\sum_{k=-\infty}^{\infty} h[k]e^{-\sigma_o k}\right| = \left|\sum_{k=0}^{\infty} h[k]e^{-\sigma_o k}\right|$$
$$\leq \sum_{k=0}^{\infty} |h[k]|\, e^{-\sigma_o k}$$
$$\leq \sum_{k=0}^{\infty} |h[k]| < \infty$$

That is, the summation converges to a real number, which we write as $H(\sigma_o)$ to show the dependence of the real number on the value chosen for σ_o. Thus the output signal is a scalar multiple of the input signal,

$$y[n] = H(\sigma_o)\, e^{\sigma_o n}$$

where

$$H(\sigma_o) = \sum_{k=0}^{\infty} h[k]\, e^{-\sigma_o k}$$

Of course the exponential input, which begins as a vanishingly small signal at $n \to -\infty$, grows without bound as *n* increases, as does the response, unless $\sigma_o = 0$ or $H(\sigma_o) = 0$.

Example For the LTI system with unit-pulse response

$$h[n] = (0.5)^n u[n]$$

given any $\sigma_o \geq 0$ we can compute

$$H(\sigma_o) = \sum_{k=0}^{\infty} (0.5)^k e^{-\sigma_o k} = \sum_{k=0}^{\infty} (e^{-\sigma_o}/2)^k = \frac{1}{1-e^{-\sigma_o}/2}$$

Therefore the response of the system to the input

$$x[n] = e^{\sigma_o n}\,, \quad -\infty < n < \infty$$

is

$$y[n] = \frac{2}{2-e^{-\sigma_o}}\, e^{\sigma_o n}$$

It is important to note that only one summation must be evaluated to compute the eigenfunction response. Contrast this with the usual convolution, which typically involves a family of summations with the nature of the summation changing with the value of *n*.

- *Complex Eigenfunctions* It is mathematically convenient to consider complex

input signals to LTI systems, though of course the unit-pulse response, *h*[*n*], is assumed to be real. If a complex input signal is written in rectangular form as

$$x[n] = x_R[n] + jx_I[n]$$

where, for each *n*,

$$x_R[n] = \mathrm{Re}\{x[n]\}\,, \quad x_I[n] = \mathrm{Im}\{x[n]\}$$

then the corresponding, complex output signal is

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$
$$= \sum_{k=-\infty}^{\infty} x_R[k]h[n-k] + j\sum_{k=-\infty}^{\infty} x_I[k]h[n-k]$$

That is,

$$\mathrm{Re}\{y[n]\} = \mathrm{Re}\{(x*h)[n]\} = (x_R * h)[n]\,,$$
$$\mathrm{Im}\{y[n]\} = \mathrm{Im}\{(x*h)[n]\} = (x_I * h)[n]$$

and so we get two real input-output calculations for a single complex calculation. (We note in passing that linearity of an LTI system holds for complex-coefficient linear combinations of complex input signals.)

As an important application of this fact, suppose the LTI system is stable, and consider the input signal

$$x[n] = e^{j\omega_o n}\ , \quad -\infty < n < \infty$$

where ω_o is a real number. The corresponding output signal is

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega_o(n-k)} = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega_o k}\ e^{j\omega_o n}$$

Since the system is stable, and $|\,e^{j\omega_o k}\,| = 1$ for every *k*,

$$\left|\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega_o k}\right| \le \sum_{k=-\infty}^{\infty} |\,h[k]e^{-j\omega_o k}\,|$$
$$\le \sum_{k=-\infty}^{\infty} |\,h[k]\,|$$
$$< \infty$$

and we have, for any frequency ω_o, convergence of the sum to a (complex) number that we write as $H(\omega_o)$. Then

$$y[n] = H(\omega_o)\, e^{j\omega_o n}$$

where

$$H(\omega_o) = \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega_o k}$$

Again, there is not a family of summations (as in convolution, in which different values of *n* can lead to different forms of the sum) to evaluate for this complex input signal, rather there is a single summation to evaluate!

Example If

$$x[n] = \cos(\omega_o n) = \mathrm{Re}\{e^{j\omega_o n}\}$$

and the system is stable, then

$$y[n] = \mathrm{Re}\{H(\omega_o)\, e^{j\omega_o n}\}$$

To evaluate this expression, write the complex number $H(\omega_o)$ in polar form as

$$H(\omega_o) = |H(\omega_o)| e^{j\sphericalangle H(\omega_o)}$$

Then

$$y[n] = \text{Re}\{|H(\omega_o)| e^{j(\omega_o n + \sphericalangle H(\omega_o))}\}$$
$$= |H(\omega_o)| \cos(\omega_o n + \sphericalangle H(\omega_o))\}$$

That is, the response of a stable system to a cosine input signal is a cosine with the same frequency but with amplitude adjustment and phase shift. Another way to write this response follows from writing $H(\omega_o)$ in rectangular form, but we leave this as an exercise.

- *Steady-State Eigenfunctions* Suppose that the system is causal as well as stable,

and that the input signal is a right-sided complex exponential,

$$x[n] = e^{j\omega_o n}\ u[n]$$

Then $y[n] = 0$ for $n < 0$, and for $n \geq 0$,

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega_o (n-k)} u[n-k]$$
$$= \sum_{k=0}^{n} h[k] e^{-j\omega_o k}\ \ e^{j\omega_o n}$$

As $n \to \infty$, remembering that the unit-pulse response is right sided by causality, and absolutely summable by stability,

$$\sum_{k=0}^{n} h[k]\, e^{-j\omega_o k} \ \to\ H(\omega_o) = \sum_{k=0}^{\infty} h[k]\, e^{-j\omega_o k}$$

and therefore, as n increases, $y[n]$ approaches the "steady-state response"

$$y_{ss}[n] = H(\omega_o) e^{j\omega_o n}$$

That is, for large values of n, $y[n] \approx y_{ss}[n]$. This is a "one-sided input, steady-state output" version of the eigenfunction property of complex exponentials. Of course a similar property for one-sided sine and cosine inputs is implied via the operations of taking real and imaginary parts.

5.6 DT LTI Systems Described by Linear Difference Equations

Systems described by constant-coefficient, linear difference equations are LTI systems. In exploring this fact, it is important to keep in mind that our default setting is that all signals are defined for $-\infty < n < \infty$. This brings about significant differences (!) with other treatments of difference equations.

Suppose we have a system whose input and output signals are related by

$$y[n] + ay[n-1] = bx[n]\,, \quad -\infty < n < \infty$$

where a and b are real constants. This is called a *first-order, constant-coefficient, linear difference equation*. Given an input signal $x[n]$, this can be viewed as an equation that must be solved for $y[n]$, and we leave to other courses the argument that for each input signal, $x[n]$, there is a unique solution for the output signal, $y[n]$. We simply make the claim that the solution is

$$y[n] = \sum_{k=-\infty}^{n} (-a)^{n-k}\, bx[k]$$

and verify this solution as follows. Using the assumed $y[n]$,

$$ay[n-1] = a\sum_{k=-\infty}^{n-1}(-a)^{n-1-k}b\, x[k]$$
$$= -\sum_{k=-\infty}^{n-1}(-a)^{n-k}b\, x[k]$$

Therefore

$$y[n] + ay[n-1] = \sum_{k=n}^{n}(-a)^{n-k}bx[k]$$
$$= bx[n]$$

Of course, the solution can be written as

$$y[n] = \sum_{k=-\infty}^{\infty}(-a)^{n-k}bu[n-k]\, x[k]$$

so it is clear that the difference equation describes an LTI system with unit-pulse response

$$h[n] = (-a)^n bu[n]$$

That this is the unit-pulse response also can be verified directly, by showing that

$$h[n] + ah[n-1] = b\delta[n]$$

From the form of $h[n]$ it follows that a first-order, constant-coefficient, linear difference equation defines a causal LTI system. Furthermore the system is memoryless if and only if $a = 0$, and stable if and only if $|a| < 1$.

Results are similar for systems described by second-order, constant-coefficient, linear difference equations,

$$y[n] + a_1\, y[n-1] + a_2\, y[n-2] = bx[n], \quad -\infty < n < \infty$$

as well as higher order. That is, such equations describe causal, LTI systems. However, it is more difficult to compute the unit-pulse response, and to connect the stability property to the coefficients in the difference equation.

Right-Sided Setting Often we are interested in right-sided inputs, where the causality property of systems described by difference equations implies that the corresponding outputs are also right sided. This means that the difference equation need only be addressed for $n \geq 0$, though we must view the output as zero for negative values of n. Therefore the "initial conditions" are $y[-1] = y[-2] = \cdots = 0$. Nonzero initial conditions, as considered in mathematics or other engineering courses dealing with difference equations, cannot arise in the context of LTI systems, for a consequence of input-output linearity is that the identically zero input signal must yield the identically zero output signal. In summary, since we are focusing on systems whose input-output behavior is linear, we must require zero initial conditions in a right-sided setting.

Exercises

1. Suppose an LTI system with input signal $x[n] = u[n] - u[n-2]$ has the response $y[n] = 2r[n] - 2r[n-2]$. Sketch this input signal and output signal, and also sketch the system response to each of the input signals below.

(a) $x_a[n] = 3u[n-1] - 3u[n-3]$

(b) $x_b[n] = u[n] - u[n-1] - u[n-2] + u[n-3]$

(c) $x_c[n] = u[n] - u[n-4]$

2. Using the graphical method, compute and sketch $y[n] = (h * x)[n]$ for

(a) $x[n] = \delta[n] - \delta[n-3]$, $h[n] = 3\delta[n+1] - 3\delta[n-3]$

(b) $x[n] = \begin{cases} 1, & 0 \le n \le 3 \\ 0, & else \end{cases}$, $h[n] = \begin{cases} 1, & 1 \le n \le 3,\ 7 \le n \le 9 \\ 0, & else \end{cases}$

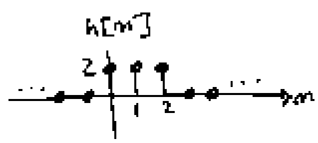
(c) $x[n] = 1$, *for all n*, $h[n] = \delta[n] - 2\delta[n-1] + \delta[n-2]$

(d) $x[n] = u[n-1] - u[n-3]$, $h[n] = -u[n] + u[n-3]$

(e) $x[n] = e^n \left(u[n] - u[n-2]\right)$, $h[n] = e^{-n}u[n]$

(f) $x[n] = u[n]$, $h[n] = (1/2)^n u[n-1]$

(g) $x[n] = r[n]$, $h[n]$ shown below



(h) $x[n] = u[n]$, $h[n] = r[n]u[3-n]$

(i) $x[n] = u[n-1] - u[n-4]$, $h[n] = (-1)^n u[n]$

(j) $x[n] = r[n]$, $h[n] = \delta[n] - 2\delta[n-1] + \delta[n+1]$

3. Using the analytical method, compute and sketch $y[n] = (h * x)[n]$ for the following signals, where α and β are distinct real numbers.

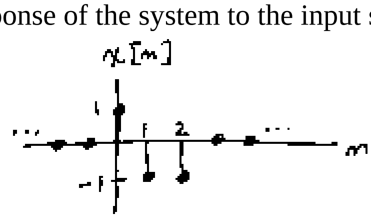
(a) $x[n] = \alpha^n u[n]$, $h[n] = \beta^n u[n]$

(b) $x[n] = \alpha^n$, $h[n] = \beta^n u[-n]$ (What additional assumption on α and β is needed?)

(c) $x[n] = \delta[n-1]$, $h[n] = 4u[3-n]$

(d) $x[n] = \alpha^{n-2}u[n-2]$, $h[n] = \beta$ (What additional assumption on α and β is needed?)

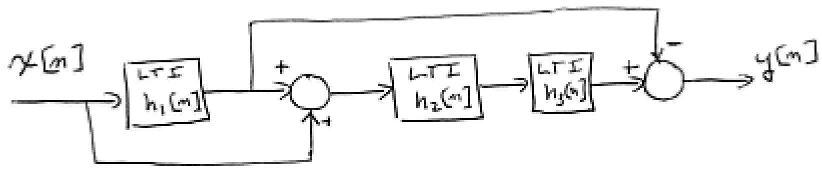
4. An LTI system with a unit-step input signal has the response $y[n] = (1/2)^n u[n]$. What is the response of the system to the input signal shown below.



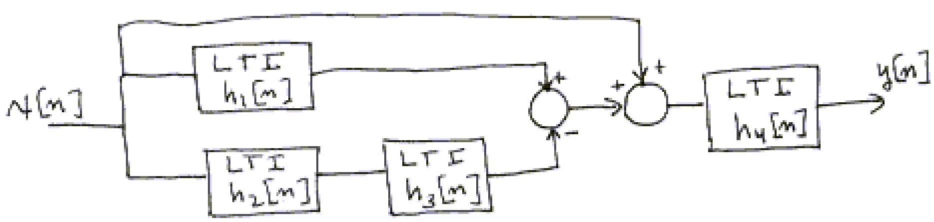
5. Consider discrete-time signals $h[n]$ that is zero outside the interval $n_0 \le n \le n_1$ and $x[n]$ that is zero outside the interval $n_2 \le n \le n_3$. Show how to define n_4 and n_5 such that $(h * x)[n] = 0$ outside the interval $n_4 \le n \le n_5$.

6. Compute the overall unit-pulse response for the interconnections of DT LTI systems shown below.

(a)



(b)



7. Suppose $y[n] = (x * h)[n]$. For each of the pairs of signals given below, show how $\hat{y}[n] = (\hat{x} * \hat{h})[n]$ is related to $y[n]$.

(a) $\hat{x}[n] = x[n-3]$, $\hat{h}[n] = h[n+3]$

(b) $\hat{x}[n] = x[n-3]$, $\hat{h}[n] = h[n-3]$

(c) $\hat{x}[n] = x[-n]$, $\hat{h}[n] = h[-n]$

(d) $\hat{x}[n] = x[-1-n]$, $\hat{h}[n] = h[1-n]$

8. Determine if the DT LTI systems with the following unit-pulse responses are causal and/or stable.

(a) $h[n] = \left(\frac{1}{2}\right)^n u[n+1]$

(b) $h[n] = \left(\frac{1}{2}\right)^n u[-n]$

(c) $h[n] = 2^n u[3-n]$

(d) $h[n] = 2^n r[-n]$

9. For the DT LTI system with unit-pulse response $h[n] = \left(\frac{1}{2}\right)^n u[n]$, use the eigenfunction properties to compute the response to the input signals

(a) $x[n] = 1$

(b) $x[n] = (-1)^n$

(c) $x[n] = 2\cos(\pi n / 2)$

(d) $x[n] = 3\sin(-\frac{3\pi}{2} n)$

6.1 CT LTI Systems and Convolution

The treatment of the continuous-time case parallels the discrete-time case, except that some facts are more difficult to prove. It is easy to check that given a continuous-time signal $h(t)$, a system described by the *convolution integral*

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

is an LTI system. (We are assuming here, as usual, that $h(t)$ and the input signal $x(t)$ are such that the integral is defined.) Indeed, combining the characterizing conditions for linearity and time invariance, we need only check the following. For any input signals $x_1(t)$ and $x_2(t)$, with corresponding responses $y_1(t)$ and $y_2(t)$, and for any constants a and t_o, the response to

$$\hat{x}(t) = a\,x_1(t) + x_2(t-t_o)$$

should be

$$\hat{y}(t) = a\,y_1(t) + y_2(t-t_o)$$

So, for a system described by convolution, we compute the response to $\hat{x}(t)$ as

$$\hat{y}(t) = \int_{-\infty}^{\infty} \hat{x}(\tau)h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} [a\,x_1(\tau) + x_2(\tau-t_o)]h(t-\tau)\,d\tau$$

$$= a\int_{-\infty}^{\infty} x_1(\tau)\,h(t-\tau)\,d\tau + \int_{-\infty}^{\infty} x_2(\tau-t_o)h(t-\tau)\,d\tau$$

Changing the variable of integration from τ to $\sigma = \tau - t_o$ in the last integral gives

$$\hat{y}(t) = a\int_{-\infty}^{\infty} x_1(\tau)\,h(t-\tau)\,d\tau + \int_{-\infty}^{\infty} x_2(\sigma)h(t-t_o-\sigma)\,d\sigma$$

$$= a\,y_1(t) + y_2(t-t_o)$$

Thus a system described by convolution is an LTI system. Furthermore, the characterizing signal $h(t)$ is the unit-impulse response of the system, as is easily verified by a sifting calculation: if $x(t) = \delta(t)$, then

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} \delta(\tau)h(t-\tau)d\tau = h(t)$$

It is also true that the input-output behavior of essentially all continuous-time, linear, time-invariant systems can be described by the convolution expression. However, to prove this fact involves delicate arguments that we will skip.

Evaluation of the Convolution Integral

Calculating the response of a system to a given input signal by evaluation of the convolution integral is not as simple as might be expected. This is because a family of integrations, parametrized by t, must be evaluated, and the character of the integration can change with the parameter. There are three main approaches.

Analytical Method When both the impulse response $h(t)$ and the input signal $x(t)$ have simple analytical descriptions, the convolution integral sometimes can be evaluated by analytical means.

Example If the input signal and the unit-impulse response are unit-step signals, then the response can be written as

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} u(\tau)u(t-\tau)d\tau$$

Of course, the integrand is zero for negative τ, regardless of the value of t, and so

$$y(t) = \int_{0}^{\infty} u(t-\tau)d\tau$$

Now the integrand is zero for $\tau > t$, but this simplification involves the value of t. In fact

$$y(t) = \begin{cases} 0\,, & t \le 0 \\ \int_0^t 1\, d\tau = t\,, & t > 0 \end{cases}$$

Summarizing, we can write the response as the unit ramp:

$$y(t) = t\,u(t) = r(t)$$

Example If the unit-impulse response is a unit-step function and the input signal is the constant signal $x(t) = 1$, then the response calculation is

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} u(t-\tau)d\tau$$

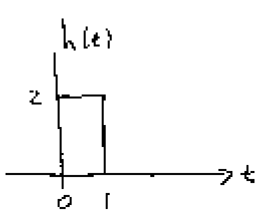
Of course the conclusion is that the response is undefined for any value of t! This is a reminder that convolution expressions must be checked to make sure they are meaningful.

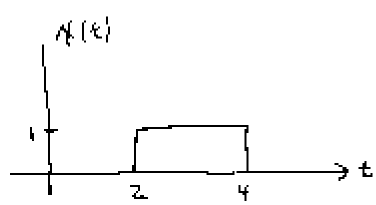
Graphical Method For more complicated cases, a graphical approach is valuable for keeping track of the calculations that must be done. Basically, we plot the two signals in the integrand, $x(\tau)$ and $h(t-\tau)$, versus τ, for the value of t of interest. Then multiplying the two signals provides the integrand, and the net area must be computed.

Example We compute

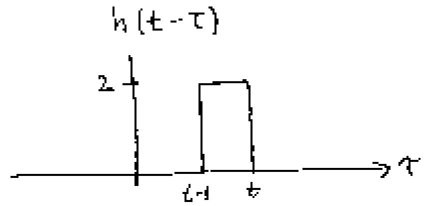
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

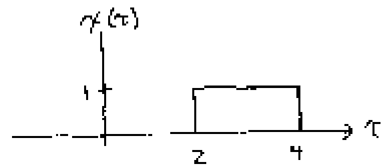
for the input signal and unit-impulse response shown below.





First the impulse response is flipped and shifted on the τ axis to a convenient value of t. Then the input signal is plotted in the variable τ immediately below to facilitate the multiplication of signals:





It is easy to see that for $t < 2$ and for $t > 5$ the product of the two signals is identically zero, and so the response is $y(t) = 0$ for these two ranges of t.

For $2 \le t \le 3$,

$$y(t) = \int_{2}^{t} 2\, d\tau = 2t - 4$$

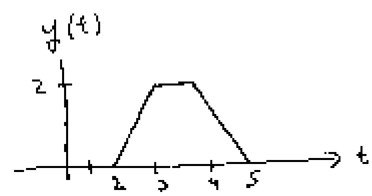
For $3 \le t \le 4$,

$$y(t) = \int_{t-1}^{t} 2\, d\tau = 2$$

For $4 < t \le 5$,

$$y(t) = \int_{t-1}^{4} 2\, d\tau = 10 - 2t$$

Of course these calculations are also obvious from consideration of the multiplication of the two signals in the various ranges and sketches of the resulting integrand. In any case, sketching the response yields



LTI Cleverness Method If one of the signals in the convolution can be written as a linear combination of simple, shifted signals, then by the properties of linearity and time invariance, the response can be computed from a single convolution involving the simple signals.

Example In the example given above, we can write $x(t) = u(t-2) - u(t-4)$, that is, $x(t)$ is a linear combination of shifted step functions. If we compute the convolution

$$\hat{y}(t) = \int_{-\infty}^{\infty} u(\tau) h(t - \tau) d\tau$$

then the response we seek is given by

$$y(t) = \hat{y}(t-2) - \hat{y}(t-4)$$

The reader is encouraged to work the details and sketch $y(t)$.

The Web lecture linked below can be consulted for further discussion.

ILM: LTI Systems and Convolution

But be aware that the notation in the Web lecture is a bit different than, and not as good as, what we have been using in class.

6.2 Properties of Convolution – Interconnections of CT LTI Systems

This topic also is covered in the interactive Web lecture. In our class notation, where we write

$$y(t) = (h * x)(t)$$

the various properties of convolution appear as follows:

- *Commutativity:*

$$(x * h)(t) = (h * x)(t)$$

- *Distributivity:*

$$\big(x * (h_1 + h_2)\big)(t) = (x * h_1)(t) + (x * h_2)(t)$$

- *Associativity:*

$$\big((x * h_1) * h_2\big)(t) = \big(x * (h_1 * h_2)\big)(t)$$

Additional properties include one that follows from linearity: For any constant *b*,

$$\big((bh) * x\big)(t) = b\,\big(h * x\big)(t)$$

Also, one that follows from time invariance, though the notation is a bit awkward: For any time t_o, if $\hat{x}(t) = x(t - t_o)$, then

$$\big(\hat{x} * h\big)(t) = \big(x * h\big)(t - t_o)$$

Finally, writing the unit impulse function as $\delta(t)$, we note that $(\delta * x)(t) = x(t)$, for any $x(t)$ continuous at $t = 0$, and furthermore $(\delta * \delta)(t) = \delta(t)$. This last expression invokes Special Property 1 of Section 2.2, and ignores continuity requirements of the sifting property. But in the present context Special Property 1 states the plainly simple fact that the unit-impulse response of the system whose unit-impulse response is a unit impulse is a unit impulse.

All of these properties can be interpreted in terms of block diagram manipulations involving LTI systems, just as in the discrete-time case.

6.3 CT LTI System Properties

The input-output behavior of a continuous-time LTI system is described by its unit-impulse response, $h(t)$, via the convolution expression

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Therefore the input-output properties of an LTI system can be characterized in terms of properties of $h(t)$. The basic results are similar to the discrete-time case, though, as usual in continuous time, there are unmentioned technical assumptions to guarantee that integrals are defined, and so on.

- *Causal System* An LTI system is causal if and only if $h(t) = 0$ for $t < 0$, that is, if and only if $h(t)$ is right sided.

Since the unit-impulse input is nonzero only at $t = 0$, and in particular is zero for $t < 0$, causality is equivalent to $h(t) = 0$ for $t < 0$. (Here we rely on the fact that the response of an LTI system to the identically-zero input signal is identically zero, and up to the time $t = 0$ a causal system does not know whether the input signal continues to be zero, or takes a nonzero value at $t = 0$.)

- *Memoryless System* An LTI system is memoryless if and only if $h(t) = 0$ for $t \neq 0$.

If $h(t) = 0$ for $t \neq 0$, then since

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

it follows that $y(t)$ can only depend on $x(t)$. On the other hand, if $h(t) \neq 0$ for $t = t_a \neq 0$, then the unit-impulse input, which is nonzero only at $t = 0$ yields a response that is nonzero at the nonzero time t_a. Thus the system is not memoryless.

Suppose $h(t)$ is nonzero at only one point in time. Then unless $h(t)$ is an impulse the response of the system to every input signal will be $y(t) = 0$ for all t. It follows from this discussion that a memoryless LTI system is characterized by an impulse response of the form $h(t) = b\,\delta(t)$, where *b* is a real constant.

- *Stable System* An LTI system is (bounded-input, bounded-output) stable if and only if the unit-impulse response is absolutely integrable. That is

$$\int_{-\infty}^{\infty} |\, h(t)\,|\, dt$$

is finite.

To prove this, suppose $x(t)$ is a bounded input, and $|\, x(t)\,| \leq M$, for all *t*. We use the fact that the absolute value of an integral with upper limit greater than lower limit is bounded by the integral of the absolute value of the integrand. This should be believable from the corresponding fact about sums. Then the absolute value of the output signal satisfies

$$|\, y(t)\,| \leq \int_{-\infty}^{\infty} |\, h(\tau)\,|\,|\, x(t-\tau)\,|\, d\tau \leq M \int_{-\infty}^{\infty} |\, h(\tau)\,|\, d\tau \leq MK$$

for all *t*, and therefore the system is stable.

To prove that stability of the system implies absolute integrability of $h(t)$, we use the same sort of cleverness as in the discrete-time case. Consider the bounded input signal

$$x(t) = \begin{cases} 1, & h(-t) \ge 0 \\ -1, & h(-t) < 0 \end{cases}$$

Then the corresponding output signal, $y(t)$, is bounded, say by the constant *K*, for all *t*. In particular, at $t = 0$,

$$K \ge y(0) = \int_{-\infty}^{\infty} h(\tau)x(-\tau)\, d\tau = \int_{-\infty}^{\infty} |\, h(\tau)\,|\, d\tau$$

Thus $h(t)$ is absolutely integrable.

- *Invertible System* First note that the identity system in continuous time, $y(t) = x(t)$, has the unit impulse response $h(t) = \delta(t)$, and the inverse system for an LTI system must be an LTI system. Then we can make the following statement: An LTI system described by $h(t)$ is invertible if and only if there exists a signal $h_I(t)$ (the impulse response of the inverse system) such that

$$(h * h_I)(t) = \delta(t)$$

Such an $h_I(t)$ might not exist, and if it does, it might be difficult to compute. We will not pursue this further.

6.4 Response to Singularity Signals

It is easy to show that the response of a CT LTI system to a unit-step input signal is the running integral of the unit-impulse response. Indeed, if $x(t) = u(t)$, then

$$y(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} h(\tau)d\tau$$

It takes a bit more work to show that the unit-ramp response can be written as an iterated running integral of the unit-impulse response. With $x(t) = r(t)$, we can write

$$y(t) = \int_{-\infty}^{\infty} h(\tau_1)r(t-\tau_1)d\tau_1 = \int_{-\infty}^{t} h(\tau_1)(t-\tau_1)d\tau_1$$

where a subscripted variable of integration has been used to make the end result pretty. Applying integration-by-parts to this expression gives

$$y(t) = (t-\tau_1)\int_{-\infty}^{\tau_1} h(\tau_2)d\tau_2 \,\Big|_{-\infty}^{t} - \int_{-\infty}^{t}\int_{-\infty}^{\tau_1} h(\tau_2)d\tau_2\ (-d\tau_1)$$

Evaluating the first term at $\tau_1 = t$ and $\tau_1 = -\infty$ shows that it vanishes, leaving

$$y(t) = \int_{-\infty}^{t}\int_{-\infty}^{\tau_1} h(\tau_2)d\tau_2\ d\tau_1$$

For an input signal that has a regular geometric shape and can be represented conveniently as a linear combination of singularity signals, these expressions can be used to write the corresponding response as a linear combination of running integrals of the unit-impulse response. Of course, the utility of such an expression depends on the complexity of $h(t)$.

Example For the input signal

$$x(t) = 2u(t) - u(t-1) + 3r(t-2)$$

the response of a CT LTI system can be written as

$$y(t) = 2\int_{-\infty}^{t} h(\tau)d\tau - \int_{-\infty}^{t-1} h(\tau)d\tau + 3\int_{-\infty}^{t-2}\int_{-\infty}^{\tau_1} h(\tau_2)d\tau_2\, d\tau_1$$

6.5 Response to Exponentials (Eigenfunction Properties)

For important classes of LTI systems, the responses to certain types of exponential input signals have particularly simple forms. These simple forms motivate many approaches to the analysis of LTI systems and we consider several variants, each of which requires slightly different assumptions on the system. As in the discrete-time case, an input signal is called an *eigenfunction* of the system if the response is simply a constant multiple of the input signal.

- *Real Eigenfunctions* Suppose the LTI system is causal and stable, and suppose

the input signal is the real, growing exponential

$$x(t) = e^{\sigma_o t}\,, \quad \sigma_o \ge 0\,, \quad -\infty < t < \infty$$

Then

$$y(t) = \int_{-\infty}^{\infty} h(\tau)e^{\sigma_o(t-\tau)}d\tau = \int_{-\infty}^{\infty} h(\tau)e^{-\sigma_o\tau}d\tau\, e^{\sigma_o t}$$

$$= \int_{0}^{\infty} h(\tau)e^{-\sigma_o\tau}d\tau\, e^{\sigma_o t}$$

where the lower limit has been raised to zero since the unit-impulse response is right sided. Convergence of the integral is guaranteed by the stability assumption, and by the fact that $|e^{-\sigma_o\tau}| \le 1\,, \quad \tau \ge 0$. The details rely on the fact that the absolute value of an integral is less than the integral of the absolute value (so long as the upper limit is greater than the lower limit. Explicitly,

$$\left|\int_{0}^{\infty} h(\tau)e^{-\sigma_o\tau}d\tau\right| \le \int_{0}^{\infty} |h(\tau)e^{-\sigma_o\tau}|\, d\tau \le \int_{0}^{\infty} |h(\tau)|\, d\tau < \infty$$

Therefore we can define the constant $H(\sigma_o)$ by

$$H(\sigma_o) = \int_{0}^{\infty} h(\tau)e^{-\sigma_o\tau}d\tau$$

and write

$$y(t) = H(\sigma_o)e^{\sigma_o t}\,, \quad -\infty < t < \infty$$

It is important to observe that only one integral must be evaluated to compute this response, in contrast with the general convolution calculation that involves a family of integrals.

- *Complex Eigenfunctions* Though we consider only real LTI systems, that is,

systems with a real unit-impulse response $h(t)$, it is convenient for mathematical purposes to permit complex-valued input signals. To see how the general calculation proceeds, write a complex input signal in rectangular form

$$x(t) = x_R(t) + jx_I(t)$$

where, for each t,

$$x_R(t) = \mathrm{Re}\{x(t)\}, \quad x_I(t) = \mathrm{Im}\{x(t)\}$$

Then, since $h(t)$ is real, and j is a constant,

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} x_R(\tau)h(t-\tau)d\tau + j\int_{-\infty}^{\infty} x_I(\tau)h(t-\tau)d\tau \end{aligned}$$

That is,

$$\mathrm{Re}\{y(t)\} = \mathrm{Re}\{(x*h)(t)\} = (x_R * h)(t)$$
$$\mathrm{Im}\{y(t)\} = \mathrm{Im}\{(x*h)(t)\} = (x_I * h)(t)$$

This means that with one complex calculation we include two real calculations

The most important application of complex inputs is the case the LTI system is stable and the input is a phasor,

$$x(t) = e^{j\omega_o t}, \quad -\infty < t < \infty$$

The response is given by

$$y(t) = \int_{-\infty}^{\infty} h(\tau)e^{j\omega_o(t-\tau)}d\tau = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega_o\tau}d\tau\; e^{j\omega_o t}$$

Here we define the complex constant

$$H(\omega_o) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega_o\tau}d\tau$$

where convergence of the integral is guaranteed by the stability assumption and the fact that

$$|e^{-j\omega_o\tau}| \le 1, \quad -\infty < \tau < \infty$$

Thus we have

$$y(t) = H(\omega_o)\, e^{j\omega_o t}, \quad -\infty < t < \infty$$

Example Suppose the signal $x(t) = \sin(\omega_o t)$, $-\infty < t < \infty$, is applied to a stable LTI system with unit-impulse response $h(t)$. Since $x(t) = \mathrm{Im}\{e^{j\omega_o t}\}$, we immediately have that $y(t) = \mathrm{Im}\{H(\omega_o)e^{j\omega_o t}\}$. This expression can be made more explicit by writing $H(\omega_o)$ in polar form:

$$H(\omega_o) = |H(\omega_o)|\, e^{j\measuredangle H(\omega_o)}$$

Then

$$\begin{aligned} y(t) &= \mathrm{Im}\{H(\omega_o)e^{j\omega_o t}\} = \mathrm{Im}\{|H(\omega_o)|\, e^{j\measuredangle H(\omega_o)}e^{j\omega_o t}\} \\ &= |H(\omega_o)|\, \mathrm{Im}\{e^{j(\omega_o t + \measuredangle H(\omega_o))}\} \\ &= |H(\omega_o)| \sin(\omega_o t + \measuredangle H(\omega_o)), \quad -\infty < t < \infty \end{aligned}$$

An alternate expression follows from writing $H(\omega_o)$ in rectangular form,

$$H(\omega_o) = \text{Re}\{H(\omega_o)\} + j\,\text{Im}\{H(\omega_o)\}$$

Then

$$y(t) = \text{Im}\left\{[\text{Re}\{H(\omega_o)\} + j\,\text{Im}\{H(\omega_o)\}][\cos(\omega_o t) + j\sin(\omega_o t)]\right\}$$
$$= \text{Re}\{H(\omega_o)\}\cos(\omega_o t) - \text{Im}\{H(\omega_o)\sin(\omega_o t)\,, \quad -\infty < t < \infty$$

Regardless of the particular form chosen for $y(t)$, the key fact is the following. If the input is a sinusoid (or phasor) of frequency ω_o, then the output is a sinusoid (or phasor) with the same frequency, although the amplitude and phase angle, relative to the input sinusoid, is altered by the system.

- *SteadyState Eigenfunctions* Suppose the LTI system is causal as well as stable,

and the input signal is the right-sided phasor

$$x(t) = e^{j\omega_o t}u(t)$$

Then $y(t) = 0$ for $t < 0$, by causality, and for $t \geq 0$,

$$y(t) = \int_{-\infty}^{\infty} h(\tau)e^{j\omega_o(t-\tau)}d\tau = \int_0^t h(\tau)e^{-j\omega_o\tau}d\tau\; e^{j\omega_o t}\,, \quad t \geq 0$$

Therefore as *t* increases, $y(t)$ more and more closely approximates the *steady-state response*

$$y_{ss}(t) = H(\omega_o)\, e^{j\omega_o t}$$

where

$$H(\omega_o) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega_o\tau}d\tau = \int_0^{\infty} h(\tau)e^{-j\omega_o\tau}d\tau$$

and again the stability assumption guarantees that $H(\omega_o)$ is well defined. Thus the steady-state response of a causal and stable LTI system to a sinusoid (or phasor) of frequency ω_o, is a sinusoid (or phasor) with the same frequency.

It is interesting to compare this steady-state property with the previous case where the phasor input signal defined for $-\infty < t < \infty$ results in the phasor output at every value of t. Of course, since the input began at $t = -\infty$, every value of t is a "steady-state" value in that an infinite period of time has elapsed since the beginning of the input signal.

6.6 CT LTI Systems Described by Linear Differential Equations

Systems described by constant-coefficient, linear differential equations are LTI systems. However, while stating this fact it is important to keep in mind that by LTI we mean "input-output linear" systems and that our default time interval is $-\infty < t < \infty$. Because ofthis setting, our treatment may not be as similar to other treatments as you might expect.

Consider a system where the input and output signals are related by

$$\dot{y}(t) + ay(t) = bx(t)\,, \quad -\infty < t < \infty$$

where *a* and *b* are real constants. This is called a *first-order, constant-coefficient, linear differential equation*. Once $x(t)$ is specified, this can be viewed as an equation that must be solved for $y(t)$. It can be shown that there is only one solution, and we will demonstrate that this solution can be written as

$$y(t) = \int_{-\infty}^{t} e^{-a(t-\tau)} bx(\tau)\, d\tau$$

The demonstration involves substituting into the differential equation, and proceeds in an elementary fashion by writing

$$y(t) = e^{-at} \int_{-\infty}^{t} e^{a\tau} bx(\tau)\, d\tau$$

In this form, the calculation of $\dot{y}(t)$ is a simple matter of the product rule, and the fundamental theorem of calculus. Indeed,

$$\begin{aligned}\dot{y}(t) &= -ae^{-at} \int_{-\infty}^{t} e^{a\tau} bx(\tau)\, d\tau + e^{-at}\, e^{at} bx(t) \\ &= -ay(t) + bx(t)\end{aligned}$$

and the solution is verified.

By inserting the appropriate unit-step function, we can write $y(t)$ in the form

$$y(t) = \int_{-\infty}^{\infty} be^{-a(t-\tau)} u(t-\tau)\, x(\tau)\, d\tau$$

and it is clear that the differential equation describes an LTI system with unit-impulse response

$$h(t) = be^{-at} u(t)$$

Remark It is interesting to show directly that this impulse response satisfies the differential equation (for all *t*) when $x(t) = \delta(t)$. The verification involves using generalized calculus to compute

$$\dot{h}(t) = -bae^{-at} u(t) + be^{-at} \delta(t) = -bae^{-at} u(t) + b\delta(t)$$

Then it is easy to see that

$$\dot{h}(t) + ah(t) = b\delta(t)\,, \quad -\infty < t < \infty$$

From the form of the unit-impulse response, $h(t)$, it follows that the LTI system described by the first-order linear differential equation is causal and is not memoryless. The system is stable if and only if $a > 0$.

For a second-order, constant-coefficient, linear differential equation,

$$\ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = bx(t)$$

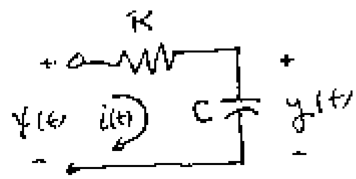
and also for higher-order linear differential equations, the situation is similar to the first-order case. Such equations describe causal LTI systems. However it is more difficult to compute the unit-impulse response, and to characterize stability properties in terms of the coefficients of the differential equation.

Right-Sided Setting In other courses you may have encountered linear differential equations defined for $t \geq 0$, with initial conditions specified at $t = 0^-$. For example, in the first-order case, consider

$$\dot{y}(t) + ay(t) = bx(t)\,, \quad t \geq 0$$

with $y(0^-)$ and $x(t), t \ge 0$, specified. This setting can be embedded into our framework by considering the input signal to be zero for $t < 0$. Then, by causality, the output signal is zero for $t < 0$, and in particular, $y(0^-)$ must be zero. (Recall that if the input signal to an LTI system is zero for all *t*, then the output signal must be zero for all *t*.) Put another way, a constant-coefficient, linear differential equation with right-sided input signals describes an LTI system if and only if all initial conditions are zero.

Example Suppose a voltage signal, $x(t)$, is applied to the terminals of a series *R*-*C* circuit shown below, and the output signal of interest, $y(t)$, is the voltage across the capacitor, *C*.



Kirchhoff's voltage law gives the circuit description as a first-order differential equation

$$\dot{y}(t) + \frac{1}{RC} y(t) = \frac{1}{RC} x(t)\,, \quad -\infty < t < \infty$$

This describes an LTI system with unit-impulse response

$$h(t) = \frac{1}{RC} e^{-\frac{1}{RC}t} u(t)$$

If we are interested in the response of this system to sinusoidal inputs with frequency ω_o, we consider the input signal

$$x(t) = e^{j\omega_o t}\,, \quad -\infty < t < \infty$$

and compute

$$\begin{aligned} H(\omega_o) &= \int_{-\infty}^{\infty} h(\tau) e^{-j\omega_o \tau}\, d\tau = \int_0^{\infty} \frac{1}{RC} e^{-\frac{1}{RC}\tau}\, e^{-j\omega_o \tau}\, d\tau \\ &= \frac{-1}{1 + jRC\omega_o} e^{-(\frac{1}{RC} + j\omega_o)\tau} \Big|_0^{\infty} \\ &= \frac{1}{1 + jRC\omega_o} \end{aligned}$$

(Notice that the implicit assumption that R and C are positive is crucial in the evaluation of the integral. This is the stability requirement – with positive R and C, $h(t)$ is absolutely integrable.) Thus the response to the phasor input signal is

$$y(t) = \frac{1}{1 + jRC\omega_o} e^{j\omega_o t}\,, \quad -\infty < t < \infty$$

From this basic fact, we can extract the response to various sinusoidal input signals. For example, if the voltage input signal is

$$x(t) = \cos(\omega_o t)\, u(t)$$

Then the steady-state response of the circuit can be written as

$$y_{ss}(t) = \text{Re}\left\{\frac{1}{1+jRC\omega_o} e^{j\omega_o t}\right\} = \text{Re}\left\{\frac{1}{\sqrt{1+R^2C^2\omega_o^2}} e^{j(\omega_o t - \tan^{-1}(RC\omega_o))}\right\}$$

$$= \frac{1}{\sqrt{1+R^2C^2\omega_o^2}} \cos[\omega_o t - \tan^{-1}(RC\omega_o)]$$

If the input frequency, ω_o, is large, then the steady-state voltage across the capacitor will be small. On the other hand, if the input frequency is small, then the steady-state response is similar in amplitude to the input signal. The phase angle of the response, relative to the input signal, also depends on the frequency. Furthermore, if the input signal is a linear combination of sinusoids at various frequencies, then the steady-state response will contain the same set of frequencies, but with the amplitudes and phase angles influenced according to $H(\omega_o)$ at the various values of ω_o. This is the basis of frequency-selective filtering.

Exercises

1. Using the graphical method, compute and sketch $y(t) = (h * x)(t)$ for

(a) $h(t) = e^{-t}u(t)$, $\quad x(t) = 2u(t) - 2u(t-1)$

(b) $h(t) = e^{-|t|}$, $\quad x(t) = u(t)$

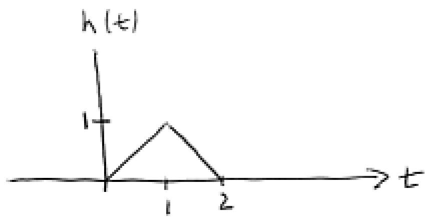
(c) $h(t) = e^{t}u(-t)$, $\quad x(t) = u(t-2)$

(d) $h(t) = e^{-t}u(t)$, $\quad x(t) = u(3-t)$

(e) $h(t) = e^{-2t}u(t)$, $\quad x(t) = e^{t-1}u(t)$

(f) $h(t) = e^{t}$, $\quad x(t) = \delta(t) - u(t)$

2. An LTI system has the impulse response shown below:



For an input signal of the form

$$x(t) = \sum_{k=0}^{\infty} a_k \delta(t - kT)$$

sketch the output signal if

(a) $T = 3$, $\quad a_k = 1$ for all $k \geq 0$

(b) $T = 2$, $\quad a_k = 1 \;\; k \geq 0$

(c) $T = 3$, $\quad a_k = (1/2)^k \;\; k \geq 0$

3. Suppose the continuous-time signal $h(t)$ is zero outside the interval $t_0 \le t \le t_1$ and the signal $x(t)$ is zero outside the interval $t_2 \le t \le t_3$. Show how to define t_4 and t_5 such that $(h * x)(t) = 0$ outside the interval $t_4 \le t \le t_5$.

4. Express $\hat{y}(t) = (\hat{h} * \hat{x})(t)$ in terms of $y(t) = (h * x)(t)$ for the following signal choices.

(a) $\hat{x}(t) = x(t-1)$, $\hat{h}(t) = h(t-2)$

(b) $\hat{x}(t) = x(t-2)$, $\hat{h}(t) = h(t+2)$

(c) $\hat{x}(t) = x(2t)$, $\hat{h}(t) = h(-2t)$

(d) $\hat{x}(t) = x(3t)$, $\hat{h}(t) = h(3t)$

(e) $\hat{x}(t) = x(-t)$, $\hat{h}(t) = h(-t)$

5. Using the analytical method, compute and sketch $y(t) = (h * x)(t)$ for

(a) $h(t) = e^t$, $x(t) = \delta(t) - u(t)$

6. Determine if the LTI systems described by the following unit-impulse responses are stable and/or causal.

(a) $h(t) = e^{-2t}u(t+3)$

(b) $h(t) = e^{3t}u(-4-t)$

(c) $h(t) = e^{-4|t|}$

(d) $h(t) = e^{t}u(t-3)$

7. Determine if the following statements about LTI systems are true or false. Justify your answers.

(a) If $h(t)$ is right sided and bounded, then the system is stable.

(b) If $h(t)$ is periodic and not identically zero, then the system is unstable.

(c) The cascade connection of a causal LTI system and a non-causal LTI system is always non-causal.

(d) A memoryless LTI system is always stable.

8. Consider a system described by

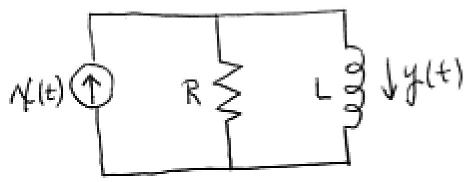
$$y(t) = \int_{-\infty}^{t} e^{-2(t-\tau)} x(\tau - 1) d\tau$$

(a) Show that this is an LTI system.

(b) Compute the unit-impulse response of the system.

(c) Compute the response of the system to $x(t) = u(t) - u(t-1)$ by convolution and then by using the fact that the unit-step response of an LTI system is the running integral of the unit-impulse response..

9. For the R-L circuit shown below, with input and output current signals as shown, and $R = 4, \quad L = 4$, compute



(a) the steady-state response to the input signal $x(t) = 3\cos(t)u(t)$.
(b) the steady-state response to $x(t) = 2\sin(3t)u(t)$.
(c) the response to $x(t) = 1$.

10. For the LTI system with unit-impulse response $h(t) = e^{t}u(t)$, compute the response to the input signal $x(t) = e^{3t}\cos(2t)$. (*Hint:* We did not discuss all the eigenfunction properties that LTI systems have.)

Notes for Signals and Systems

7.1 Introduction to CT Signal Representation

A fundamental idea in signal analysis is to represent signals in terms of linear combinations of 'basis' signals. That is, we choose a set of basis signals,

$$\phi_0(t), \phi_1(t), \ldots, \phi_{K-1}(t)$$

that are relatively simple, have useful properties, and are well suited to the class of signals to be represented. Then, given $x(t)$, we compute scalar coefficients $a_0, a_1, \ldots, a_{K-1}$ such that

$$x(t) \approx a_0\, \phi_0(t) + a_1\, \phi_1(t) + \cdots + a_{K-1}\, \phi_{K-1}(t)$$

There are many reasons for this approach. Certainly it makes sense for signal processing by LTI systems, particularly if the basis signals have nice properties as input signals to such systems. Also, storage or transmission of a signal can be accomplished by storage or transmission of the coefficients, $a_0, a_1, \ldots, a_{K-1}$, once a set of basis signals has been selected.

There are many basic questions to be addressed in developing this approach: What properties of basis sets would be useful? How many basis signals are needed? What is the appropriate nature of the approximation "$\approx$?" Answers to these questions are developed in some detail in the next few sections. Those wishing to omit the general discussion can proceed directly to Section 8.1 where the representation of main interest is introduced in an ad-hoc fashion.

Some examples can motivate the discussion.

Example Consider the signal

$$x(t) = \begin{cases} e^{-t}, & -1 \le t \le 1 \\ 0, & else \end{cases}$$

and the basis set of three signals that are zero outside the interval $-1 \le t \le 1$, with

$$\phi_0(t) = 1, \phi_1(t) = t, \phi_2(t) = t^2/2; \quad -1 \le t \le 1$$

(Indeed, it would be simpler to dispense with our default domain of definition of signals and simply work on the interval $-1 \le t \le 1$ as the domain of definition. We retain the default mainly for emphasis.)

Recalling Taylor's formula, we can choose

$$a_0 = x(0) = 1, a_1 = \dot{x}(0) = -1, a_2 = \ddot{x}(0) = 1$$

to obtain the representation

$$x(t) \approx \phi_0(t) - \phi_1(t) + \phi_2(t) = \begin{cases} 1 - t + t^2/2, & -1 \le t \le 1 \\ 0, & else \end{cases}$$

Thus Taylor's formula fits the framework we are considering, and we have some notion of the sense of approximation. That is, the approximation will be good for values of *t* close to $t = 0$, and exact only at zero. Of course the class of signals to be represented in this way must be twice differentiable at the $t = 0$, but the computation of the coefficients is rather simple. Furthermore, if we want to refine the representation by adding additional basis signals, for example $\phi_3(t) = t^3/(3!)$, it is obvious how to compute the coefficient a_3, though the signal must be thrice differentiable at $t = 0$. An advantage of the setup is that in computing the fourth coefficient, first three coefficients in the representation do not change.

Example Consider the same signal and basis set, but suppose we now require that the approximation have zero error at the three values $t = -1, 0, 1$. Recalling polynomial interpolation, we proceed by setting

$$x(t) = a_0\,\phi_0(t) + a_1\,\phi_1(t) + a_2\,\phi_2(t), \quad t = -1, 0, 1$$

This yields three equations in three unknowns:

$$e = a_0 - a_1 + a_2 / 2$$

$$1 = a_0$$

$$e^{-1} = a_0 + a_1 + a_2 / 2$$

Solving this set of equations gives

$$a_0 = 1, \quad a_1 = \frac{1 - e^2}{2e} = -1.186, \quad a_2 = \frac{e^2 - 2e + 1}{e} = 1.063$$

and the resulting representation is

$$x(t) \approx \phi_0(t) - 1.186\phi_1(t) + 1.063\phi_2(t)$$

$$= \begin{cases} 1 - 1.186t + 0.503t^2, & -1 \le t \le 1 \\ 0, & else \end{cases}$$

This representation is perhaps better for some purposes than the Taylor's formula, though it is inconvenient to have to solve equations for the coefficients. Also, if a fourth basis signal is added to the set, say $\phi_3(t) = t^3 / 3!$, and we require zero error at a fourth point to be consistent with polynomial interpolation capabilities, then the representation must be recomputed from the beginning as the values of the first three coefficients will change.

7.2 Orthogonality and Minimum ISE Representation

A popular choice of the nature of the approximation in signal representation, and the approximation we focus on in the sequel, is the following. Given an energy signal $x(t)$ and a set of basis energy signals $\phi_0(t), \phi_1(t), \ldots, \phi_{K-1}(t)$, suppose the coefficients $a_0, a_1, \ldots, a_{K-1}$ are computed to minimize the *integral square error*,

$$I = \int_{-\infty}^{\infty} \left(x(t) - \sum_{k=0}^{K-1} a_k \phi_k(t) \right)^2 dt$$

Because of the square, the representation error at each point in time is positively weighted in the process of minimization, and larger errors are more severely penalized. Both of these features are sensible for signal representation.

- **Orthogonality** Partly to ease the problem of computing the minimizing coefficients, we require that the basis set satisfy the condition

$$\int_{-\infty}^{\infty} \phi_l(t)\,\phi_k(t)\,dt = 0; \quad l \ne k, \quad k = 0, \ldots, K-1$$

A basis set that satisfies this condition is called *orthogonal*, and one consequence of orthogonality is that when the integrand in *I* is expanded, a large number of cross-terms disappear. It is notationally convenient to denote the energy of the k^{th} basis signal by

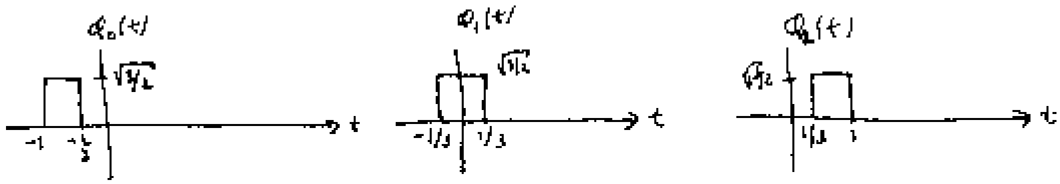
$$E_k = \int_{-\infty}^{\infty} \phi_k^2(t)\,dt, \quad k = 0, 1, \ldots, K-1$$

If all the E_k -coefficients are unity in an orthogonal basis set, the basis set is called *orthonormal*. It is important to emphasize that orthogonality and orthomormality are properties of basis *sets*. If a new basis signal is added to an orthogonal set, the check that the new set is orthogonal involves all the signals in the set. Also, if the time interval of interest is changed, the orthogonality condition might not hold for the new interval.

Example The basis set used in the examples in Section 7.1, $\phi_0(t) = 1, \phi_1(t) = t, \phi_2(t) = t^2/2; \quad -1 \le t \le 1$, with all signals zero outside this interval, is not orthogonal since, for example,

$$\int_{-\infty}^{\infty} \phi_0(t)\phi_2(t)\,dt = \int_{-1}^{1} t^2/2\ dt \neq 0$$

On the other hand, the basis set of rectangular pulses



is orthogonal. This is clear because the basis signals are non-overlapping in the obvious sense. Also, the basis set is orthonormal since each $\phi_k^2(t)$ is a unit-area rectangular pulse. (Of course, overlapping signals also can be orthogonal, and orthonormal, but it is usually not obvious!)

- **Minimum ISE**

Given $x(t)$ and an orthogonal basis set

$$\phi_0(t), \phi_1(t), \ldots, \phi_{K-1}(t)$$

with basis signal energies

$$E_0, E_1, \ldots, E_{K-1}$$

the minimum ISE coefficients are given by

$$a_k = \frac{1}{E_k}\int_{-\infty}^{\infty} x(t)\phi_k(t)\,dt = 0, \quad k = 0, \ldots, K-1$$

To prove this result, first expand the quadratic integrand in *I* and distribute the integral over the sum of terms. Using orthogonality and the notation for basis signal energy, this gives

$$I = \int_{-\infty}^{\infty} x^2(t)\ dt - 2\sum_{k=0}^{K-1}\int_{-\infty}^{\infty} x(t)\phi_k(t)\,dt\ a_k + \sum_{k=0}^{K-1} E_k a_k^2$$

To minimize *I*, set the derivative of *I* with respect to each coefficient a_k to zero,

$$\frac{\partial I}{\partial a_k} = 0, \quad k = 0, 1, \ldots, K-1$$

Because I is a quadratic polynomial in the coefficients, this is quite easy, yielding

$$-2\int_{-\infty}^{\infty} x(t)\,\phi_k(t)\,dt + 2E_k a_k = 0, \quad k = 0, \ldots, K-1$$

This expression easily rearranges to the claimed formula for the coefficients. To show that this indeed provides a minimum, it can be shown that the matrix of second partials is positive definite, an easy exercise left to the reader.

Remark Orthogonality is such a useful property of basis sets that non-orthogonal sets are seldom encountered. Indeed, families of orthogonal basis sets of very different natures, suitable for representing widely varying classes of signals, have been discovered and cataloged. These sets often are named after the discoverer.

Example Consider again the signal

$$x(t) = e^{-t}, \quad -1 \le t \le 1$$

where we abandon the artifice of defining the given signal, and the basis signals, to be zero for $|t| > 1$, and simply work on the specified interval. This time we choose the first three *Legendre* basis signals:

$$\phi_0(t) = 1, \quad \phi_1(t) = t, \quad \phi_2(t) = \tfrac{3}{2}t^2 - \tfrac{1}{2}; \quad -1 \le t \le 1$$

and leave verification of orthogonality as an exercise, as well as verification that

$$E_0 = 2, \quad E_1 = \frac{2}{3}, \quad E_2 = \frac{2}{7}$$

The minimum integral-squared-error representation is specified by the coefficients

$$a_0 = \tfrac{1}{2}\int_{-1}^{1} x(t)\phi_0(t)\,dt = \tfrac{1}{2}\int_{-1}^{1} e^{-t}\,dt = \tfrac{e-e^{-1}}{2} = 1.18$$

$$a_1 = \tfrac{3}{2}\int_{-1}^{1} x(t)\phi_1(t)\,dt = \tfrac{3}{2}\int_{-1}^{1} te^{-t}\,dt = -3e^{-1} = -1.10$$

$$a_2 = \tfrac{7}{2}\int_{-1}^{1} x(t)\phi_2(t)\,dt = \tfrac{7}{2}\int_{-1}^{1} (\tfrac{3}{2}t^2 - \tfrac{1}{2})e^{-t}\,dt = \frac{e^2 - 7}{e} = 0.14$$

yielding the representation

$$x(t) \approx 1.18\phi_0(t) - 1.10\phi_1(t) + 0.14\phi_2(t)$$

$$= 1.18 - 1.10t + 0.14(\tfrac{3t^2}{2} - \tfrac{1}{2}), \quad -1 \le t \le 1$$

where the approximation is understood to be minimum integral-squared error using the first three Legendre basis signals. This representation can be refined by adding additional basis signals, and if orthogonality of the basis set is preserved we need only compute the new coefficients. For example, the fourth Legendre basis signal is

$$\phi_3(t) = (5/2)t^3 - (3/2)t$$

with $E_3 = 2/7$. Then, skipping the actual evaluation of the coefficient

$$a_3 = \frac{2}{7}\int_{-1}^{1} x(t)\phi_3(t)\,dt$$

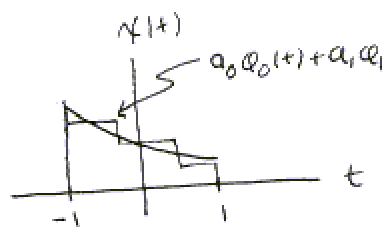
the resulting representation is

$$x(t) \approx 1.18\,\phi_0(t) - 1.10\,\phi_1(t) + 0.14\,\phi_2(t) + a_3\,\phi_3(t)$$
$$= 1.18 - 1.10\,t + 0.14\,(\tfrac{3t^2}{2} - \tfrac{1}{2}) + a_3\,(\tfrac{5t^3}{2} - \tfrac{3t}{2})\,, \quad -1 \le t \le 1$$

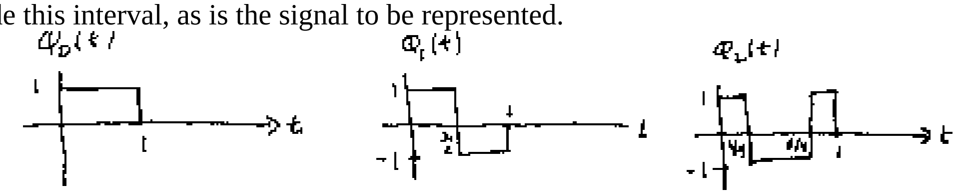
Example We could also represent the signal in the preceding example using the orthonormal basis of rectangular pulses considered before. In this case the minimum integral-squared error coefficients are given by

$$a_0 = \int_{-\infty}^{\infty} x(t)\,\phi_0(t)\,dt = \int_{-1}^{-1/3} e^{-t}\sqrt{3/2}\,dt = \sqrt{3/2}\,(e - e^{1/3})$$
$$a_1 = \int_{-\infty}^{\infty} x(t)\,\phi_1(t)\,dt = \int_{-1/3}^{1/3} e^{-t}\sqrt{3/2}\,dt = \sqrt{3/2}\,(e^{1/3} - e^{-1/3})$$
$$a_2 = \int_{-\infty}^{\infty} x(t)\,\phi_2(t)\,dt = \int_{1/3}^{1} e^{-t}\sqrt{3/2}\,dt = \sqrt{3/2}\,(e^{-1/3} - e^{-1})$$

The nature of the resulting representation is shown below, and it is clear that this basis set is not particularly well suited to smoothly varying signals. Also, note that in this case it is not clear how to add basis signals to maintain orthonormality and improve the representation.



Example Consider the first three basis signals in the *Walsh* basis set, as shown below. These typically are defined on the time interval $0 \le t \le 1$, and we assume the basis signals are zero outside this interval, as is the signal to be represented.



It is straightforward to verify that this is an orthonormal basis set by time slicing the integrals involved. For example,

$$\int_0^1 \phi_0(t)\phi_1(t)\,dt = \int_0^{1/2} 1\,dt + \int_{1/2}^{1} (-1)\,dt = 0$$

Using this basis set to represent the signal

$$x(t) = e^{-t}, \quad 0 \le t \le 1$$

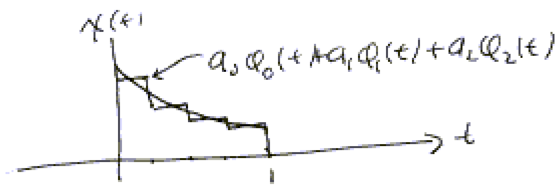
yields the coefficients

$$a_0 = \int_0^1 e^{-t}\, dt = 1 - e^{-1} = 0.632$$

$$a_1 = \int_0^{1/2} e^{-t}\, dt - \int_{1/2}^{1} e^{-t}\, dt = 1 - 2e^{-1/2} + e^{-1} = 0.155$$

$$a_2 = \int_0^{1/4} e^{-t}\, dt - \int_{1/4}^{3/4} e^{-t}\, dt - \int_{3/4}^{1} e^{-t}\, dt = 0.019$$

and the representation shown below.



In this case there is a natural continuation of the basis set that will refine the staircase approximation evident in the representation, though we leave further details to references.

7.3 Complex Basis Signals

Even though we are interested in representing real signals, it turns out that complex basis signals can be mathematically convenient. A suitable notation for a complex basis set is to number the basis signals as

$$\phi_{-K}(t), \phi_{-(K-1)}(t), \ldots, \phi_{-1}(t), \phi_0(t), \phi_1(t), \ldots, \phi_K(t)$$

where $\phi_0(t)$ is real, and

$$\phi_{-k}(t) = \phi_k^*(t)\,, \quad k = 1, 2, \ldots K$$

As we will see below, the condition that conjugate basis signals be included in the set yields the pleasing result that the approximation to a real signal is real. These $2K+1$ basis signals should be considered on the same interval as the signals to be represented, with everything set to zero outside this interval. We simply choose the default interval, $-\infty < t < \infty$, for the purpose of exposition.

The appropriate definition of integral-squared error must account for the possibility of a complex representation, though, as mentioned above, this will not occur. Thus we use the magnitude squared, instead of the square, in the integrand, and write

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \left| x(t) - \sum_{k=-K}^{K} a_k \phi_k(t) \right|^2 dt \\ &= \int_{-\infty}^{\infty} \left(x(t) - \sum_{k=-K}^{K} a_k \phi_k(t) \right) \left(x(t) - \sum_{k=-K}^{K} a_k \phi_k(t) \right)^* dt \\ &= \int_{-\infty}^{\infty} \left(x(t) - \sum_{k=-K}^{K} a_k \phi_k(t) \right) \left(x(t) - \sum_{k=-K}^{K} a_k^* \phi_k^*(t) \right) dt \end{aligned}$$

It turns out that the appropriate definition of orthogonality in this case is the condition

$$\int_{-\infty}^{\infty} \phi_l(t)\,\phi_k^*(t)\,dt = 0\,, \quad l \neq k$$

While we will not justify it in detail, it is clear that this condition eliminates cross-terms among the basis signals in expanding the quadratic integrand of *I*. Also, we let

$$E_k = \int_{-\infty}^{\infty} |\phi_k(t)|^2\,dt = \int_{-\infty}^{\infty} \phi_k(t)\,\phi_k^*(t)\,dt\,, \quad k = 0, \pm 1, \ldots, \pm K$$

With this setup, it can be shown that the coefficients $a_{-K}, \ldots, a_K$ that minimize the integral-squared error for a real signal $x(t)$ are given by

$$a_k = \frac{1}{E_k} \int_{-\infty}^{\infty} x(t)\,\phi_k^*(t)\,dt \quad , \quad k = 0, \pm 1, \ldots, \pm K$$

Remarks

- The denominator of this expression, the real, nonnegative number E_k, typically is pre-computed for standard basis sets.
- Since $x(t)$ is real, and E_k is real, the complex conjugate of the k^{th} coefficient satisfies

$$\begin{aligned} a_k^* &= \frac{1}{E_k}\left(\int_{-\infty}^{\infty} [x(t)\,\phi_k(t)]\;\;dt\right)^* = \frac{1}{E_k}\int_{-\infty}^{\infty} [x(t)\,\phi_k^*(t)]^*\;dt \\ &= \frac{1}{E_k}\int_{-\infty}^{\infty} x(t)\,\phi_k(t)\,dt = \frac{1}{E_k}\int_{-\infty}^{\infty} x(t)\,\phi_{-k}^*(t)\,dt \\ &= \quad a_{-k} \end{aligned}$$

Therefore, only $K+1$ coefficients, not $2K+1$, need to be computed explicitly. Furthermore, this property implies that the complex conjugate of each term in the representation also is in the representation. Specifically,

$$a_{-k}\phi_{-k}(t) = a_k^*\phi_k^*(t) = [a_k\phi_k(t)]^*$$

That is, the minimum integral-square-error approximation of a real signal is a real signal, and it is perfectly sensible to write

$$x(t) \approx \sum_{k=-K}^{K} a_k\phi_k(t)$$

where the approximation is in the sense of minimum integral-square error.

7.4 DT Signal Representation

Just as in the continuous-time case, it is convenient to represent discrete-time signals as linear combinations of specified basis signals,

$$\phi_0[n], \phi_1[n], \ldots, \phi_M[n]$$

where we assume that the signal to be represented and all the basis signals are defined on the same sample range, with the default range $-\infty < n < \infty$. Though we are interested in representing real signals, sometimes complex basis signals again are mathematically convenient, and we require that conjugates be included. That is, if $\phi_k[n]$ is complex, then for some m we have $\phi_m[n] = \phi_k^*[n]$ for all n. However, in the discrete-time case it is not traditional to adopt the same numbering system as the continuous-time case.

A meaningful objective is to choose coefficients $a_0, a_1, \ldots, a_M$ to minimize the sum-squared error

$$I = \sum_{n=-\infty}^{\infty} | x[n] - \sum_{m=0}^{M} a_m \phi_m[n] |^2$$

(Magnitude signs are used for the summand because at this point we have not specified that the representation be real, though the signal $x(t)$ is assumed to be real. Once this important condition is addressed, then magnitude signs are superfluous.)

The analysis of this problem proceeds in a manner very similar to the continuous-time case. A very convenient property of basis sets is *orthogonality*, which in the discrete-time case is defined by the condition

$$\sum_{n=-\infty}^{\infty} \phi_k[n]\phi_m^*[n] = 0\,, \quad k \neq m$$

Furthermore, the basis set is called *orthonormal* if the following condition also holds:

$$\sum_{n=-\infty}^{\infty} \phi_m[n]\phi_m^*[n] = 1\,, \quad m = 0, 1, \ldots, M$$

Clearly this quantity must be a positive number, assuming that none of the basis signals is identically zero, and the condition that the positive number be unity is indeed a normalization of the basis signals.

To minimize *I* it is convenient to write the (possibly) complex coefficients a_m in rectangular form, expand the expression for *I*, and set the derivatives with respect to each real and imaginary part to zero. Orthogonality simplifies this process considerably, and the result is that the minimizing coefficients are given by

$$a_m = \frac{\sum_{n=-\infty}^{\infty} x[n]\phi_m^*[n]}{\sum_{n=-\infty}^{\infty} \phi_m[n]\phi_m^*[n]}\quad, m = 0, 1, \ldots, M$$

This expression should appear natural from the continuous-time case Further computation, also omitted, shows that the second-derivative test for minimality is satisfied.

Since we require that the basis set be self conjugate, that is for each *k* there is an *m* such that $\phi_k^*[n] = \phi_m[n]$, then the corresponding coefficients in the minimum *SSE* representation satisfy

$$a_m^* = \frac{\sum_{n=-\infty}^{\infty} x[n]\phi_m[n]}{\sum_{n=-\infty}^{\infty} \phi_m[n]\phi_m^*[n]} = \frac{\sum_{n=-\infty}^{\infty} x[n]\phi_k^*[n]}{\sum_{n=-\infty}^{\infty} \phi_k^*[n]\phi_k[n]} = a_k$$

The resulting representation

$$a_0\phi_0[n] + a_1\phi_1[n] + \cdots + a_M\phi_M[n]$$
is a real signal since for every *k* there is an *m* such that the corresponding summands satisfy
$$a_k^*\phi_k^*[n] = a_m\phi_m[n]$$
and the sums of these pairs of terms is real.

Exercises

1. Show that the basis set that is defined on $-1 \le t \le 1$ by
$$\phi_0(t) = 1\,, \quad \phi_1(t) = t\,, \quad \phi_2(t) = \tfrac{3}{2}\,t^2 - \tfrac{1}{2}$$
with the signals zero outside of this interval, is an orthogonal basis set on the interval $-\infty < t < \infty$. If we change the third signal to $\phi_2(t) = t^2\,, \quad -1 \le t \le 1$, is the new basis set orthogonal on $-\infty < t < \infty$.

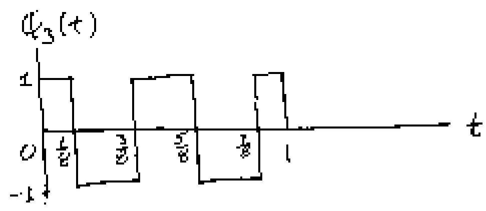
2. (a) Show that for any $x(t)$,
$$x_{ev}(t) = Ev\{x(t)\}\,, \quad x_{od}(t) = Od\{x(t)\}$$
form an orthogonal basis set over any interval of the form $-T \le t \le T$.

(b) Suppose $\phi_k(t)\,, \quad k = 0,\ldots,K-1$ is an orthogonal basis set on the interval $-1 \le t \le 1$. Let $\varphi_k(t) = \phi_k(t/3)\,, \quad k = 0,\ldots,K-1$. On what interval, $-T \le t \le T$, if any, is $\varphi_k(t)\,, \quad k = 0,\ldots,K-1$ an orthogonal set?

(c) Suppose $\phi_k(t)\,, \quad k = 0,\ldots,K-1$ is an orthogonal basis set on the interval $0 \le t \le 1$. Let $\varphi_k(t) = \phi_k(3t-1)\,, \quad k = 0,\ldots,K-1$. On what interval time interval, if any, is $\varphi_k(t)\,, \quad k = 0,\ldots,K-1$ an orthogonal set?

3. Consider the set of signals defined as shown for $0 \le t \le 4$, and defined to be zero outside this interval:
$$\phi_0(t) = \sin(\tfrac{\pi}{2}t)\,, \quad \phi_1(t) = u(t-1) - u(t-3)\,, \quad \phi_2(t) = r(t) - 2r(t-2)$$
Is this an orthogonal set on the interval $0 \le t \le 4$?

4. The fourth Walsh basis signal is defined as shown below



For the signal $x(t) = e^{-t}$, $0 \le t \le 1$, compute and sketch the
(a) minimum integral square error representation using $\phi_0(t), \phi_1(t)$,
(b) minimum integral square error representation using $\phi_2(t), \phi_3(t)$,
(c) minimum integral square error representation using the first 4 Walsh basis signals.

5. Determine values of the coefficients *a*, *b*, and *c* so that the signals

$$\phi_0(t) = ae^{-t} \;, \quad \phi_1(t) = be^{-t} + ce^{-2t}$$

form an orthonormal basis set on the time interval $0 \le t < \infty$.

Notes for Signals and Systems

8.1 CT Fourier Series

Informally, the Fourier series representation involves writing a periodic signal as a linear combination of harmonically-related sinusoids. This is a surprising yet familiar notion. For an introduction based on audio signals, visit

Listen to Fourier Series

We offer two approaches to developing the subject mathematically. For those who have skipped over the general introduction to signal representation in Chapter 7, we provide a shortcut. For those interested in a deeper understanding based on the notions in Chapter 7, we present the Fourier series as a special case of an orthogonal representation using a particular set of complex basis signals.

- **Shortcut** It is often more convenient to represent a periodic signal as a linear combination of harmonically-related complex exponentials, rather than trigonometric functions. In these terms, the basic fact is that a real, periodic signal $x(t)$, with fundamental period T_o, can be written as

$$x(t) = \sum_{k=-\infty}^{\infty} X_k \, e^{jk\omega_o t}$$

where ω_o is the fundamental frequency of $x(t)$, $\omega_o = 2\pi / T_o$. The coefficients X_k in general are complex. To see how to determine these coefficients, multiply both sides by the complex exponential signal $e^{-jl\omega_o t}$, where l is an integer and then integrate over one period. This gives, for any value of t_1,

$$\int_{t_1}^{t_1+T_o} x(t)e^{-jl\omega_o t} \, dt = \sum_{k=-\infty}^{\infty} X_k \int_{t_1}^{t_1+T_o} e^{j(k-l)\omega_o t} \, dt$$

Using the easily-verified fact that

$$\int_{t_1}^{t_1+T_o} e^{j(k-l)\omega_o t} \, dt = \begin{cases} 0, & k \neq l \\ T_o, & k = l \end{cases}$$

we obtain

$$X_l = \frac{1}{T_o} \int_{t_1}^{t_1+T_o} x(t)e^{-jl\omega_o t} \, dt$$

This shortcut provides a formula for the Fourier series coefficients of a periodic signal, though a number of issues and questions are left aside. One of these is the nature of convergence of the infinite series, and another is the nature of approximation when a truncated series is used:

$$x(t) \approx \sum_{k=-K}^{K} X_k \, e^{jk\omega_o t}$$

We note here only that the approximation is real, since the coefficients obey the conjugacy relation $X_l^* = X_{-l}$ and thus the complex congugate of the $k = l$ term is the $k = -l$ term, which is included in the sum.

- **The Fourier Basis Set** From a mathematical viewpoint, the Fourier series representation for real, periodic signals can be based on minimum integral-square-error representation using a complex, orthogonal basis set, where the basis signals are periodic with the same period as the signal being represented. This complex-form Fourier series is at first less intuitive than real forms, but it offers significant mathematical advantages.

Suppose $x(t)$ is real and periodic with fundamental period T_o and fundamental frequency $\omega_o = 2\pi / T_o$. We then choose a basis set of harmonically related phasors according to

$$\phi_k(t) = e^{jk\omega_o t} \ , \quad k = 0, \pm 1, \ldots, \pm K$$

This basis set has several properties.

- The basis set is self-conjugate, since $\phi_0(t) = 1$ and, for nonzero *k*,

$$\phi_k^*(t) = \left(e^{jk\omega_o t} \right)^* = e^{-jk\omega_o t} = e^{j(-k)\omega_o t} = \phi_{-k}(t)$$

(As in the exponent in this calculation, we often move negative signs around for convenience of interpretation.)

- Every basis signal has period T_o,

$$\begin{aligned} \phi_k(t + T_o) &= e^{jk\omega_o (t+T_o)} = e^{jk\omega_o t} \ e^{jk\frac{2\pi}{T_o}T_o} = \phi_k(t) e^{jk2\pi} \\ &= \phi_k(t) \end{aligned}$$

Furthermore, the signals $\phi_1(t)$ and $\phi_{-1}(t)$ have fundamental period T_o, the signals $\phi_2(t)$ and $\phi_{-2}(t)$ have fundamental period $T_o / 2$, and so on.

- The basis set is orthogonal on any time interval of length T_o. To show this, we compute, for $l \neq k$, and any t_1,

$$\begin{aligned} \int_{t_1}^{t_1+T_o} \phi_l(t)\phi_k^*(t)\, dt &= \int_{t_1}^{t_1+T_o} e^{jl\omega_o t} \ e^{-jk\omega_o t} \ dt = \int_{t_1}^{t_1+T_o} e^{j(l-k)\omega_o t} \ dt \\ &= \frac{1}{j(l-k)\omega_o} \, e^{j(l-k)\omega_o t} \Bigg|_{t_1}^{t_1+T_o} \\ &= \frac{e^{j(l-k)\omega_o t_1}}{j(l-k)\omega_o} \left[e^{j(l-k)\omega_o T_o} - 1 \right] \\ &= \frac{e^{j(l-k)\omega_o t_1}}{j(l-k)\omega_o} \left[e^{j(l-k)2\pi} - 1 \right] \\ &= 0 \end{aligned}$$

Also, for $l = k$,

$$E_k = \int_{t_1}^{t_1+T_o} \phi_k(t)\phi_k^*(t)\,dt = \int_{t_1}^{t_1+T_o} e^{jk\omega_o t}\, e^{-jk\omega_o t}\,dt$$
$$= \int_{t_1}^{t_1+T_o} 1\,dt = T_o$$
so that the basis set is orthonormal when $T_o = 1$.

Using these properties, we can compute the minimum integral-square-error representation for $x(t)$ over one fundamental period. Then since the signal and every term in the representation repeat, we have the minimum integral-square-error representation over any number of fundamental periods, and in fact over the interval $-\infty < t < \infty$. Of course, if there is nonzero integral square error over one period, then there will be infinite integral square error over the infinite interval. However, it is clear that by minimizing integral square error over one period we are minimizing integral square error over the interval $-\infty < t < \infty$ in a reasonable sense.

Using the general formula for coefficients that minimize integral square error, and adopting a special notation for the coefficients, let
$$X_k = \frac{1}{T_o}\int_{t_1}^{t_1+T_o} x(t)\phi_k^*(t)\,dt = \frac{1}{T_o}\int_{t_1}^{t_1+T_o} x(t)e^{-jk\omega_o t}\,dt$$
This gives the representation
$$x(t) \approx \sum_{k=-K}^{K} X_k\phi_k(t) = \sum_{k=-K}^{K} X_k\, e^{jk\omega_o t}$$
where the approximation is understood to be in the sense of minimum integral square error using $2K+1$ basis signals. Notice again that the representation is real, since, for any integer *k*, and any t_1,
$$X_k^* = \left(\frac{1}{T_o}\int_{t_1}^{t_1+T_o} x(t)\phi_k^*(t)\,dt\right)^* = \frac{1}{T_o}\int_{t_1}^{t_1+T_o} [x(t)\phi_k^*(t)]^*\,dt$$
$$= \frac{1}{T_o}\int_{t_1}^{t_1+T_o} x^*(t)\phi_k^{**}(t)\,dt = \frac{1}{T_o}\int_{t_1}^{t_1+T_o} x(t)\phi_k(t)\,dt$$
$$= \frac{1}{T_o}\int_{t_1}^{t_1+T_o} x(t)\phi_{-k}^*(t)\,dt = X_{-k}$$
Along with the relation $\phi_k^*(t) = \phi_{-k}(t)$, this implies that the complex conjugate of each term, $X_k e^{jk\omega_o t}$, in the representation is also included in the representation.

Example The periodic signal shown below is a repeated version of a signal considered in earlier examples:

Clearly $T_o = 3$, and therefore $\omega_o = 2\pi/3$. This information specifies the Fourier basis set, and the coefficients are given by

$$X_k = \frac{1}{T_o}\int_{-1}^{-1+T_o} x(t)\, e^{-jk\omega_o t}\, dt = \frac{1}{3}\int_{-1}^{2} x(t)\, e^{-jk\frac{2\pi}{3}t}\, dt$$

$$= \frac{1}{3}\int_{-1}^{1} e^{-(1+jk\frac{2\pi}{3})t}\, dt = \frac{-1}{3(1+jk\frac{2\pi}{3})}\, e^{-(1+jk\frac{2\pi}{3})t}\Big|_{-1}^{1}$$

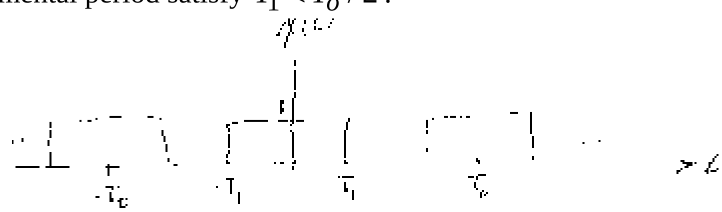
$$= \frac{e^{1+jk\frac{2\pi}{3}} - e^{-(1+jk\frac{2\pi}{3})}}{3(1+jk\frac{2\pi}{3})}$$

These coefficients are used in the expression

$$x(t) = \sum_{k=-\infty}^{\infty} X_k\, e^{jk\omega_o t}$$

We should not expect that the Fourier series coefficients will be simple or pretty in all cases! However, in some examples a little addition work yields an improved formula.

Example Consider the periodic rectangular-pulse signal shown below, where of course the pulse width and fundamental period satisfy $T_1 < T_o/2$.



The value of T_o fixes the fundamental frequency $\omega_o = 2\pi/T_o$, and also fixes the basis signals. Choosing to integrate over the fundamental period centered at the origin, that is, compute the minimum ISE representation on the one-period time interval $-T_o/2 < t \le T_o/2$, the coefficient formula gives

$$X_k = \frac{1}{T_o}\int_{-T_o/2}^{T_o/2} x(t)\phi_k^*(t)\, dt = \frac{1}{T_o}\int_{-T_1}^{T_1} e^{-jk\omega_o t}\, dt$$

It is necessary to separate out the $k = 0$ case, to obtain

$$X_0 = \frac{1}{T_o}\int_{-T_1}^{T_1} 1\, dt = \frac{2T_1}{T_o}$$

Then, for $k \ne 0$,

$$X_k = \frac{1}{T_o}\int_{-T_1}^{T_1} e^{-jk\omega_o t}\,dt = \frac{1}{T_o}\frac{-1}{jk\omega_o}e^{-jk\omega_o t}\Big|_{-T_1}^{T_1} = \frac{-1}{jk\omega_o T_o}\left(e^{-jk\omega_o T_1} - e^{jk\omega_o T_1}\right)$$
$$= \frac{1}{k\pi}\left(\frac{e^{jk\omega_o T_1} - e^{-jk\omega_o T_1}}{2j}\right) = \frac{\sin(k\omega_o T_1)}{k\pi}$$
$$= \frac{2\sin(k\omega_o T_1)}{k\omega_o T_o}$$

This can also be expressed in terms of the fundamental period, T_o, as

$$X_k = \frac{\sin(k2\pi T_1 / T_o)}{k\pi}$$

It turns out that this expression for the coefficients usually is presented in terms of a standard mathematical function, $\text{sinc}(\theta)$, defined by

$$\text{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$$

Minor rewriting of X_k gives, for $k \neq 0$,

$$X_k = \frac{\sin(k2\pi T_1 / T_o)}{k\pi} = \frac{2T_1}{T_o}\frac{\sin(\pi 2kT_1 / T_o)}{\pi 2kT_1 / T_o} = \frac{2T_1}{T_o}\text{sinc}(k\frac{2T_1}{T_o})$$

Furthermore, a simple application of L'Hospital's rule shows that $\text{sinc}(0) = 1$. Therefore this expression for the coefficients is suitable for all values of *k*. Thus we can write

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{2T_1}{T_o}\text{sinc}(k\frac{2T_1}{T_o})\, e^{jk\frac{2\pi}{T_o}t}$$

Alternately, again, we can write this representation in terms of the fundamental frequency as

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{\omega_o T_1}{\pi}\text{sinc}(k\frac{\omega_o T_1}{\pi})\, e^{jk\omega_o t}$$

In any case, as a matter of good style, Fourier series coefficients that are in fact real should always be manipulated into real form!

8.2 Real Forms, Spectra, and Convergence

- **Real Forms** The complex-form Fourier series,

$$x(t) = \sum_{k=-K}^{K} X_k e^{jk\omega_o t}$$

can be rewritten in at least three different ways. First, we can group complex-conjugate terms to write

$$x(t) = X_0 + \sum_{k=1}^{K}[X_k e^{jk\omega_o t} + X_{-k}e^{-jk\omega_o t}]$$
$$= X_0 + \sum_{k=1}^{K}[X_k e^{jk\omega_o t} + (X_k e^{jk\omega_o t})^*]$$
$$= X_0 + \sum_{k=1}^{K} 2\,\text{Re}\{X_k e^{jk\omega_o t}\}$$

Expressing X_k in polar form as $X_k = |X_k| e^{j\measuredangle X_k}$ yields

$$\begin{aligned} x(t) &= X_0 + \sum_{k=1}^{K} 2\,\mathrm{Re}\{|X_k| e^{j(k\omega_o t + \measuredangle X_k)}\} \\ &= X_0 + \sum_{k=1}^{K} 2\,|X_k| \cos(k\omega_o t + \measuredangle X_k) \end{aligned}$$

This is called the *cosine trigonometric form*.

If we write X_k in rectangular form as $X_k = \mathrm{Re}\{X_k\} + j\,\mathrm{Im}\{X_k\}$, then

$$\begin{aligned} x(t) &= X_0 + \sum_{k=1}^{K} 2\,\mathrm{Re}\{X_k e^{jk\omega_o t}\} \\ &= X_0 + \sum_{k=1}^{K} 2\,\mathrm{Re}\left\{(\mathrm{Re}\{X_k\} + j\,\mathrm{Im}\{X_k\}) e^{jk\omega_o t}\right\} \\ &= X_0 + \sum_{k=1}^{K} 2[\mathrm{Re}\{X_k\}\cos(k\omega_o t) - \mathrm{Im}\{X_k\}\sin(k\omega_o t)] \end{aligned}$$

This is called the *sine-cosine form* of the Fourier series.

A third form follows by again writing X_k in polar form. Using the facts that X_0 is real, and the real-part of a sum is the sum of the real parts, yields

$$x(t) = \mathrm{Re}\left\{ X_0 + \sum_{k=1}^{K} 2\,|X_k| e^{j(k\omega_o t + \measuredangle X_k)} \right\}$$

That is, $x(t)$ is expressed as the real part of a sum of harmonic phasors.

Each of these forms is useful in particular situations, but the complex-form Fourier series offers a general mathematical convenience and economy that is especially useful for electrical engineers. Therefore we will focus on this form, though alternate interpretations of the topics we discuss are available for the other forms as well.

- **Spectra** The *magnitude spectrum* of a T_o − periodic signal $x(t)$ is a line chart showing $|X_k|$ vs. *k* on a real axis, or, more often, vs. $k\omega_o$ on a real frequency ω axis. The *phase spectrum* of $x(t)$ is a line chart showing $\measuredangle X_k$ vs. *k* on a real axis, or vs. $k\omega_o$ on a real frequency ω axis. When the signal is such that X_k is real for all *k*, we often simply plot a line chart of X_k, and this is referred to as an *amplitude spectrum*.

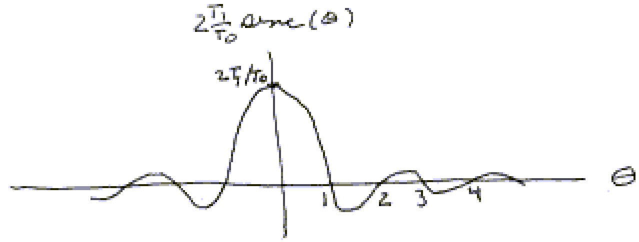
Example For the periodic rectangular pulse signal, we computed the Fourier series coefficients

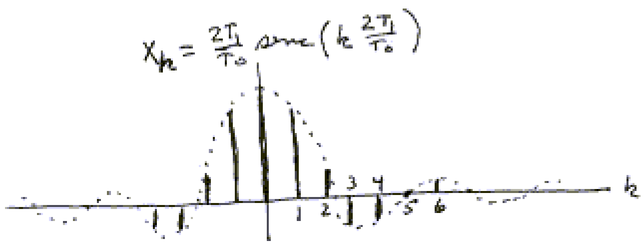
$$X_k = \frac{2T_1}{T_o}\,\mathrm{sinc}(k\frac{2T_1}{T_o}) = \frac{2T_1}{T_o}\,\mathrm{sinc}(\theta)\Big|_{\theta = k\frac{2T_1}{T_o}}$$

where the *sinc* function is defined as

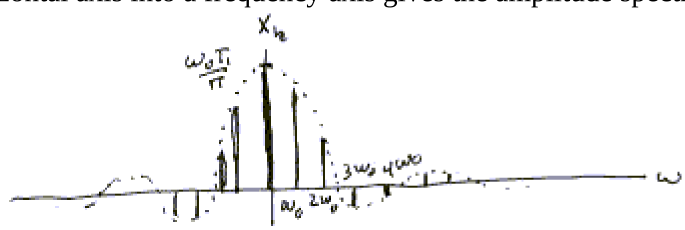
$$\mathrm{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$$

This function provides an envelope for the Fourier series coefficients, and leads to an amplitude spectrum for the rectangular pulse train. First we show a plot of the envelope $(2T_1/T_o)\,\text{sinc}(\theta)$, and then a plot of the values of the X_k using this envelope, where of course $2T_1/T_o < 1$.

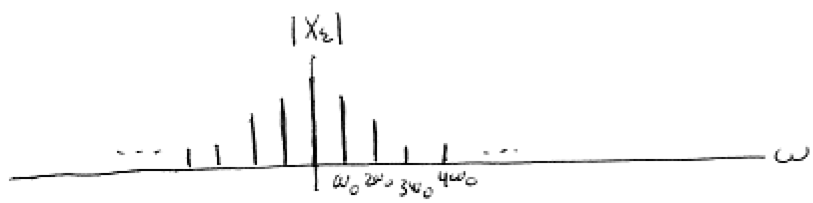


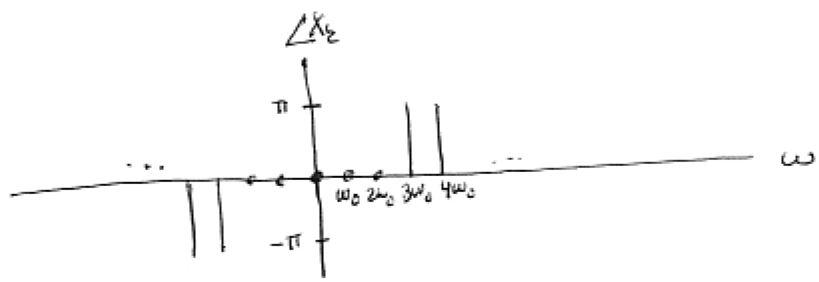


Translating the horizontal axis into a frequency axis gives the amplitude spectrum:



From this amplitude spectrum we can easily plot the magnitude and phase spectra. Since $|X_{-k}| = |X_k^*| = |X_k|$, the magnitude spectrum is symmetric about the vertical axis, and since $\measuredangle X_{-k} = \measuredangle X_k^* = -\measuredangle X_k$, the phase spectrum can be chosen to be anti-symmetric about the vertical axis by choosing the angle range from $-\pi$ to π.





- **Convergence Issues** We have shown that the coefficient choice

$$X_k = \frac{1}{T_o} \int_{T_o} x(t) e^{-jk\omega_o t} \, dt \,, \quad k = 0, \pm 1, \pm 2, \ldots, K$$

minimizes the integral square error

$$I_{2K+1} = \int_{T_o} [x(t) - \sum_{k=-K}^{K} X_k e^{jk\omega_o t}]^2 dt$$

where we have added a subscript to emphasize that this is the integral square error with $2K+1$ basis signals. The convergence issue we address is whether the integral square error approaches zero as *K* increases. That is, under what conditions do we have

$$\lim_{K\to\infty}(I_{2K+1}) = 0$$

This is a difficult question, and we will simply state the best known sufficient condition, called the *Dirichlet condition*:

Theorem If $x(t)$ is periodic with fundamental period T_o, and if

(a) $\int_{T_o} |\, x(t) \,|\, dt < \infty$,

(b) $x(t)$ has at most a finite number of maxima and minima in one period,

(c) $x(t)$ has at most a finite number of finite discontinuities in one period,

then

(i) $\lim_{K\to\infty}(I_{2K+1}) = 0$,

(ii) at each value of *t* where $x(t)$ is continuous, $x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_o t}$,

(iii) at each value of *t* where $x(t)$ has a discontinuity, $\sum_{k=-\infty}^{\infty} X_k e^{jk\omega_o t}$ takes the value of the mid-point of the discontinuity.

Notice that the Dirichlet condition is a sufficient condition for a type of convergence of the Fourier series, and this is not the type of convergence typically studied in beginning calculus. Because it is a sufficient condition, there are signals that do not satisfy the Dirichlet condition but nonetheless have Fourier series with similar convergence properties. Also, there are different types of convergence that can be considered, though we will not consider these issues further.

The nature of convergence of Fourier series results in an important phenomenon called the Gibbs Effect when a truncated (finite) Fourier series is used as an approximation to the signal. For details on this, and on the important notion of windowing the coefficients to remove the effect, see the Web lecture

Harmonic Phasors and Fourier Series

and also the demonstration

Phasor Phactory

Both of these use the phasor representation of the Fourier series.

Convergence issues aside, it is remarkable how well a few terms of the Fourier series can approximate a periodic signal. To get an appreciation of this, consult the demonstration

Fourier Series Approximation

This demonstration uses the cosine-trigonometric form of the Fourier series, and an obvious, appropriate modification of the notions of magnitude and phase spectra. Sketch in various signals and notice how 4 or 5 harmonics of the Fourier series can render a good approximation.

8.3 Fourier Series Interpretations of Operations on Signals

Periodic signals are determined, to desired accuracy in terms of integral square error, by knowledge of the fundamental frequency, ω_o, and a suitable number of the complex-form Fourier series coefficients, X_k, $\quad k = 0, \pm 1, \ldots, \pm K$. Thus the time-domain view of periodic signals is complemented by a "frequency domain" view, namely, the coefficients of various harmonic frequencies that make up the signal. This raises the possibility of performing or interpreting operations on signals by performing or interpreting operations on the frequency domain representation, that is, on the Fourier series coefficients

We will not go through a long list of operations, since this topic will reappear when we consider a more general frequency-domain viewpoint that includes aperiodic signals as well. However we consider a few examples.

Example Given a signal

$$x(t) = \sum_{k=-K}^{K} X_k e^{jk\omega_o t}$$

suppose a new signal is formed by amplitude transformation, $\hat{x}(t) = ax(t) + b$, where $a \neq 0$ and b are real constants. It is clear that $\hat{x}(t)$ is periodic, with the same fundamental period/frequency as $x(t)$, and indeed it is easy to determine the Fourier series coefficients of $\hat{x}(t)$ by inspection. We simply write

$$\hat{x}(t) = \sum_{k=-K}^{K} \hat{X}_k e^{jk\omega_o t} = b + a \sum_{k=-K}^{K} X_k e^{jk\omega_o t}$$

and conclude that

$$\hat{X}_k = \begin{cases} aX_0 + b\,, & k = 0 \\ aX_k\,, & k \neq 0 \end{cases}$$

This approach relies on the fact that the terms in a Fourier series for a periodic signal are unique, a fact that should be clear since each coefficient is determined independently of the others. However, a safer approach, especially for more complicated operations, is to begin with the expression for the Fourier series coefficients of the new signal, and relate it to the expression for coefficients of the original signal.

Example Suppose $\hat{x}(t) = x(at)$, where *a* is a nonzero constant. Then $\hat{x}(t)$ is periodic with fundamental period $\hat{T}_o = T_o / |a|$ and fundamental frequency $\hat{\omega}_o = |a|\,\omega_o$. The complex-form Fourier series coefficients are given by

$$\hat{X}_k = \frac{1}{\hat{T}_o} \int_0^{\hat{T}_o} \hat{x}(t) e^{-jk\hat{\omega}_o t}\, dt = \frac{|a|}{T_o} \int_0^{T_o/|a|} x(at) e^{-jk|a|\omega_o t}\, dt$$

To proceed, we need to separate the cases of positive and negative *a*. If $a < 0$, that is, $a = -|a|$, then the change of integration variable from *t* to $\tau = at = -|a|\,t$ gives

$$\hat{X}_k = \frac{|a|}{T_o} \int_0^{-T_o} x(\tau) e^{-jk|a|\omega_o \tau / a}\, \frac{d\tau}{-|a|} = \frac{1}{T_o} \int_{-T_o}^{0} x(\tau) e^{jk\omega_o \tau}\, d\tau$$

$$= \frac{1}{T_o} \int_{-T_o}^{0} x(\tau) \left(e^{-jk\omega_o \tau} \right)^* d\tau = X_k^* = X_{-k}$$

It is left as an exercise to show that for $a > 0$ a somewhat different result is obtained, namely

$$\hat{X}_k = X_k$$

That is, time scale by a positive constant leaves the Fourier series coefficients unchanged, though of course the fundamental frequency is changed. On the other hand, as a particular example, time scale by $a = -1$, which is time reversal, leaves the fundamental frequency unchanged, and the magnitude spectrum of the signal unchanged, but changes the phase spectrum.

8.4 CT LTI Frequency Response and Filtering

We can combine the Fourier series representation for periodic signals with the eigenfunction property for stable LTI systems to represent system responses to periodic input signals. Suppose the system has unit-impulse response $h(t)$. Since we will be considering different frequencies, it is convenient to change our earlier notation. Rather than think of a fixed frequency, ω_o, we think of frequency as a variable, ω, and define the *frequency response function* of the system by

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t}\, dt$$

Of course, the stability assumption guarantees that $H(\omega)$ is well defined for all ω. Then given a periodic input signal described, at least approximately, by the Fourier series expression

$$x(t) = \sum_{k=-K}^{K} X_k e^{jk\omega_o t}$$

linearity and the eigenfunction property give the output expression

$$y(t) = \sum_{k=-K}^{K} H(k\omega_o) X_k\, e^{jk\omega_o t}$$

Letting $Y_k = H(k\omega_o)X_k$, and noting the conjugacy property that $H(-\omega) = H^*(\omega)$, we see that the Y_k coefficients satisfy $Y_{-k} = Y_k^*$. This leads to the conclusion that the Y_k's are Fourier series coefficients for the periodic output signal. This expression for $y(t)$ can be converted to various real forms in the usual way. Of course, the output signal typically has the same fundamental frequency as the input signal, though not always since the frequency response function can be zero at particular frequencies.

This property also carries over to the case of causal, stable LTI systems with "right-sided periodic" input signals. Namely, the steady-state response is periodic and is as described above.

We can consider the magnitude of the frequency response function as a frequency-dependent gain of the system. That is, $|H(k\omega_o)|$ is the gain of the system at frequency $k\omega_o$, the factor by which the amplitude of the k^{th} harmonic of the input signal is increased or decreased. This is the basis of frequency selective filtering, where an LTI system is designed to have desired effects on the frequency components of the input signal. To show the filtering properties of a system, we often display a plot of the magnitude of the frequency response function, $|H(\omega)|$, vs. ω.

Example Consider again the *R*-*C* circuit in Section 6.6, with $R = 1, C = 1$. The unit-impulse response is

$$h(t) = \frac{1}{RC} e^{-\frac{1}{RC}t} u(t) = e^{-t} u(t)$$

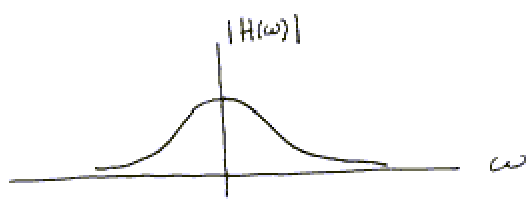
Therefore the frequency response function for the circuit is

$$H(\omega) = \int_{-\infty}^{\infty} e^{-t} u(t) e^{-j\omega t} dt = \int_{0}^{\infty} e^{-(1+j\omega)t} dt$$
$$= \frac{1}{1+j\omega}$$

Since

$$|H(\omega)| = \frac{1}{\sqrt{1+\omega^2}}$$

it is straightforward to sketch the magnitude of the frequency response function:



Clearly the circuit acts as a low-pass filter, and high-frequency input signal components are attenuated much more than low-frequency components. To be specific, suppose

$$x(t) = 1 + \cos(t) + \cos(30t)$$

Since

$$x(t) = \text{Re}\{e^{j0t}\} + \text{Re}\{e^{jt}\} + \text{Re}\{e^{j30t}\}$$

linearity and the eigenfunction property can be used to write the response as

$$y(t) = \text{Re}\{H(0)e^{j0t}\} + \text{Re}\{H(1)e^{jt}\} + \text{Re}\{H(30)e^{j30t}\}$$

Then the computations

$$H(0)=1, \quad H(1)=\tfrac{1}{\sqrt{2}}e^{-j\pi/4}, \; H(30)\approx \tfrac{1}{30}e^{-j\pi/2}$$

give

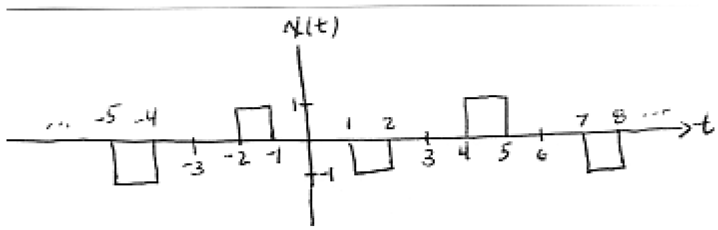
$$y(t)\approx 1+\tfrac{1}{\sqrt{2}}\cos(t-\pi/4)+\tfrac{1}{30}\cos(30t-\pi/2)$$

Exercises

1. Compute the complex-form Fourier series coefficients and sketch the magnitude and phase spectra for

(a) the signal $x(t)$ that has fundamental period $T_o = 1$, with $x(t)=e^{-t}$, $\quad 0\le t\le 1$.

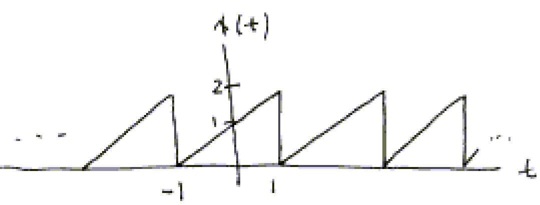
(b) the signal $x(t)$ shown below



(c) the signal

$$x(t)=\sum_{k=-\infty}^{\infty}(-1)^k\,\delta(t-2k)$$

(d) the signal $x(t)$ shown below



2. Suppose $x(t)$ is periodic with fundamental period T_o and complex-form Fourier series coefficients X_k . Show that

(a) if $x(t)$ is odd, $x(t)=-x(-t)$, then $X_k=-X_{-k}$ for all *k*.

(b) if $x(t)$ is "half-wave odd," $x(t)=-x(t+T_o/2)$, then $X_k=0$ for every even integer *k*.

(c) if $x(t)$ is even, $x(t)=x(-t)$, then $X_k=X_{-k}$ for all *k*.

3. Suppose the signal $x(t)$ has fundamental period T_o and complex-form Fourier series coefficients X_k . Derive expressions for the complex-form Fourier series coefficients of the following signals in terms of X_k .

(a) $\hat{x}(t)=2x(t-3)+1$

(b) $\hat{x}(t)=x(1-t)$

(c) $\hat{x}(t) = \frac{d}{dt} x(t)$

(d) $\hat{x}(t) = \int\limits_{-\infty}^{t} x(\tau)\, d\tau$ (What additional assumption is required on the Fourier series coefficients of $x(t)$?) *Hint:* $\hat{X}_k = \frac{1}{T_o} \int\limits_0^{T_o} \int\limits_{-\infty}^{t} x(\tau) d\tau\, e^{-jk\omega_o t}\, dt$ and integration-by-parts can be used to write this in a way that X_k can be recognized.

(e) $\hat{x}(t) = x(t + T_o / 2)$

4. Given the LTI system with unit-impulse response $h(t) = e^{-4\,|t|}$, compute the Fourier series representation for the response $y(t)$ of the system to the input signal

(a) $x(t) = \sum\limits_{n=-\infty}^{\infty} \delta(t-n)$

(b) $x(t) = \sum\limits_{n=-\infty}^{\infty} (-1)^n \delta(t-n)$

5. A continuous-time periodic signal $x(t)$ has Fourier series coefficients

$$X_k = \begin{cases} (6/\,jk)e^{jk\pi/4}\,, & k = \pm 1, \pm 3 \\ 0\,, & else \end{cases}$$

Compute and sketch the magnitude and phase spectra of the signal.

6. Answer, with justification, the following questions about the response of the stable LTI system with frequency response function

$$H(\omega) = \frac{5}{3 + j\omega}$$

(a) For a periodic input signal $x(t)$ that has fundamental period $T_o = 2\pi$, what harmonics will appear in the output signal $y(t)$ with diminished magnitude? That is, what values of *k* yield $|Y_k| < |X_k|$?

(b) For a periodic input signal that has fundamental period $T_o = \pi$, what harmonics will appear in the output signal with diminished magnitude?

7. Answer, with justification, the following questions about the response of the LTI system that has impulse response $h(t) = 25\,t\,e^{-3t}u(t)$.

(a) For a periodic input signal $x(t)$ that has fundamental period $T_o = 2$, what harmonics will appear in the output signal $y(t)$ with diminished magnitude? That is, what values of *k* yield $|Y_k| < |X_k|$?

(b) For a periodic input signal that has fundamental period $T_o = 4\pi$, what harmonics will appear in the output signal with diminished magnitude?

8. Consider the LTI system that has impulse response $h(t) = \delta(t) - e^{-2t}u(t)$, and suppose the input signal is $x(t) = 1 + 2\cos(t) + 3\cos(2t)$. Compute the response $y(t)$.

9. Consider the LTI system that has impulse response $h(t) = te^{-t}u(t) + 2e^{-t}u(t) - \delta(t)$, and suppose the input signal is $x(t) = 2 + 2\cos(t)$. Compute the response $y(t)$.

Notes for Signals and Systems

9.1 Periodic DT Signal Representation (Fourier Series)

Suppose $x[n]$ is a real, periodic signal with fundamental period N_o and, of course, fundamental frequency $\omega_o = 2\pi / N_o$. We choose a basis set of N_o harmonically related discrete-time phasors, called the *discrete-time Fourier basis set*:

$$\phi_k[n] = e^{jk\omega_o n}\ , \quad k = 0, 1, \ldots, N_o - 1$$

(Often these signals are written out in the form $e^{jk\frac{2\pi}{N_o}n}$, but we make use of the fundamental frequency to simplify the appearance of the exponent.)

This basis set has several properties:

- There are exactly N_o distinct basis signals of this type (not an infinite number as in the continuous-time case) since

$$\begin{aligned}\phi_{k+N_o}[n] &= e^{j(k+N_o)\omega_o n} = e^{jk\omega_o n}\ e^{jN_o\omega_o n} \\ &= \phi_k[n]\end{aligned}$$

- The basis set is self-conjugate. Of course, $\phi_o[n]$ is real, and for any other *k* in the range, since $e^{jN_o\omega_o} = e^{j2\pi} = 1$,

$$\begin{aligned}\phi_k^*[n] &= e^{-jk\omega_o n} = e^{-jk\omega_o n} e^{jN_o\omega_o n} = e^{j(N_o-k)\omega_o n} \\ &= \phi_{N_o-k}[n]\end{aligned}$$

- Each basis signal is periodic with period (not necessarily fundamental period) N_o,

$$\begin{aligned}\phi_k[n+N_o] &= e^{jk\omega_o(n+N_o)} = e^{jk\omega_o n}\ e^{jk2\pi} \\ &= \phi_k[n]\end{aligned}$$

- The basis set is orthogonal over any range of length N_o in *n*. To show this, consider the range $r \le n \le r + N_o - 1$, where *r* is any integer, and compute

$$\sum_{n=r}^{r+N_o-1} \phi_k[n]\phi_m^*[n] = \sum_{n=r}^{r+N_o-1} e^{jk\omega_o n} e^{-jm\omega_o n}$$

Changing the summation index from *n* to $l = n - r$ gives

$$\begin{aligned}\sum_{n=r}^{r+N_o-1} \phi_k[n]\phi_m^*[n] &= \sum_{l=0}^{N_o-1} e^{jk\omega_o(l+r)} e^{-jm\omega_o(l+r)} \\ &= e^{j(k-m)\omega_o r} \sum_{l=0}^{N_o-1} e^{j(k-m)\omega_o l}\end{aligned}$$

Next we apply the identity, which holds for any complex number α,

$$\sum_{l=0}^{N_o-1} \alpha^l = \begin{cases} N_o\ , & \alpha = 1 \\ \frac{1-\alpha^{N_o}}{1-\alpha}\ , & \alpha \ne 1 \end{cases}$$

to the summation to obtain

$$\sum_{n=r}^{r+N_o-1} \phi_k[n]\phi_m^*[n] = e^{j(k-m)\omega_o r} \sum_{l=0}^{N_o-1} \left(e^{j(k-m)\omega_o}\right)^l$$

$$= \begin{cases} N_o\,, & k=m \\ e^{j(k-m)\omega_o r}\,\frac{1-e^{j(k-m)\omega_o N_o}}{1-e^{j(k-m)\omega_o}}\,, & k \neq m \end{cases}$$

$$= \begin{cases} N_o\,, & k=m \\ 0\,, & k \neq m \end{cases}$$

Thus we have established orthogonality, and furthermore,

$$\sum_{n=r}^{r+N_o-1} \phi_m[n]\phi_m^*[n] = N_o$$

From these properties and the general formulas for minimum sum-squared-error representations in Section 7.4, we can conclude that to represent the N_o-periodic, real signal $x[n]$ with minimum sum squared error per period using the Fourier basis set, the coefficients are given by

$$X_m = \frac{1}{N_o} \sum_{n=r}^{r+N_o-1} x[n]\,\phi_m^*[n] = \frac{1}{N_o} \sum_{n=r}^{r+N_o-1} x[n]\,e^{-jm\omega_o n}$$

Here *r* is any integer, and we again use the special notation X_m to denote the m^{th} Fourier series coefficient for the signal $x[n]$. Therefore the representation is written as

$$x[n] \approx \sum_{m=0}^{N_o-1} X_m e^{jm\omega_o n}$$

where the approximation is in the sense of minimum sum-squared error. Of course it is important to immediately note that, in addition to the conjugacy relation

$$\phi_m^*[n] = e^{-jm\omega_o n} = e^{-jm\omega_o n} e^{jN_o\omega_o n} = e^{-j(N_o-m)\omega_o n}$$

$$= \phi_{N_o-m}[n]$$

we have

$$X_m^* = \frac{1}{N_o} \sum_{n=r}^{r+N_o-1} x[n]\,e^{jm\omega_o n} = \frac{1}{N_o} \sum_{n=r}^{r+N_o-1} x[n]\,e^{jm\omega_o n} e^{-jN_o\omega_o n}$$

$$= \frac{1}{N_o} \sum_{n=r}^{r+N_o-1} x[n]\,e^{-j(N_o-m)\omega_o n} = X_{N_o-m}$$

Therefore the conjugate of every term $X_m e^{jm\omega_o n}$ is included as another term in the sum, and so the minimum sum-squared-error representation is real.

Next, before working examples, we will compute a surprising expression for the minimum value of the sum squared error per period. Begin by writing the representation as

$$\sum_{m=0}^{N_o-1} X_m e^{jm\omega_o n} = \sum_{m=0}^{N_o-1} \left[\frac{1}{N_o} \sum_{l=0}^{N_o-1} x[l]\, e^{-jm\omega_o l} \right] e^{jm\omega_o n}$$

We can interchange the order of summation to write

$$\sum_{m=0}^{N_o-1} X_m e^{jm\omega_o n} = \sum_{l=0}^{N_o-1} \frac{1}{N_o} x[l] \sum_{m=0}^{N_o-1} e^{-jl\omega_o m} e^{jn\omega_o m}$$

$$= \frac{1}{N_o} \sum_{l=0}^{N_o-1} x[l] \sum_{m=0}^{N_o-1} \phi_l^*[m]\phi_n[m]$$

But

$$\sum_{m=0}^{N_o-1} \phi_l^*[m]\phi_n[m] = \begin{cases} 0, & n \ne l \\ N_o, & n = l \end{cases}$$

by orthogonality, and thus

$$\sum_{m=0}^{N_o-1} X_m e^{jm\omega_o n} = x[n]$$

That is, the sum squared error per period is zero! This means that our approximate representation actually is an *exact* representation.

Example This example is almost too simple, and there is danger of confusion, but it illustrates the calculation of the discrete-time Fourier series. Consider

$$x[n] = (-1)^n$$

In this case, $N_o = 2$, $\omega_o = \pi$, and the two basis signals are $e^{j0\pi n} = 1$ and $e^{j\pi n}$, which of course is $(-1)^n$. The Fourier coefficients are given by

$$X_0 = \frac{1}{2} \sum_{n=0}^{1} (-1)^n = 0$$

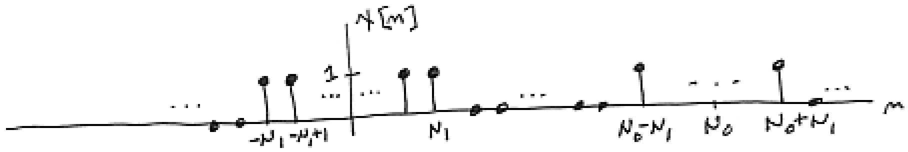
$$X_1 = \frac{1}{2} \sum_{n=0}^{1} (-1)^n e^{-j\pi n} = \frac{1 - e^{-j\pi}}{2} = 1$$

Thus the Fourier series representation is

$$x[n] = \sum_{m=0}^{1} X_m e^{jm\pi n} = \left[0 + e^{j\pi n} \right] = e^{j\pi n}$$

Certainly this is no surprise.

Example Consider a periodic train of clumps of $2N_1 + 1$ pulses of unit height repeating with fundamental period N_o, as shown below.



The Fourier coefficients can be computed from

$$X_m = \frac{1}{N_o} \sum_{n=-N_1}^{N_o - N_1 - 1} x[n] e^{-jm\omega_o n} = \frac{1}{N_o} \sum_{n=-N_1}^{N_1} e^{-jm\omega_o n}$$

Clearly $X_0 = (2N_1 + 1)/N_o$, and for nonzero *m* we replace the summation index *n* by $k = n + N_1$ to write

$$X_m = \frac{1}{N_o} \sum_{k=0}^{2N_1} e^{-jm\omega_o (k - N_1)} = \frac{1}{N_o} e^{jm\omega_o N_1} \sum_{k=0}^{2N_1} e^{-jm\omega_o k}$$

Recognizing the right-most summation to be of the form

$$\sum_{k=0}^{2N_1} \left(e^{-jm\omega_o} \right)^k = \frac{1 - e^{-jm\omega_o (2N_1 + 1)}}{1 - e^{-jm\omega_o}}$$

gives

$$\begin{aligned} X_m &= \frac{1}{N_o} e^{jm\omega_o N_1} \frac{1 - e^{-jm\omega_o (2N_1+1)}}{1 - e^{-jm\omega_o}} \\ &= \frac{1}{N_o} \frac{e^{-jm\omega_o/2} (e^{jm\omega_o N_1} e^{jm\omega_o/2} - e^{-jm\omega_o N_1} e^{-jm\omega_o/2})}{e^{-jm\omega_o/2} (e^{jm\omega_o/2} - e^{-jm\omega_o/2})} \\ &= \frac{1}{N_o} \frac{\sin(m\omega_o (N_1 + 1/2))}{\sin(m\omega_o/2)} \end{aligned}$$

(As usual, when a coefficient can be written in real form we continue computing until we obtain a real expression.) Now an easy calculation with L'Hospital's rule shows that this formula is valid for $m = 0$ as well.

This expression of the Fourier coefficients is often written in terms of evaluations of an envelope function as

$$X_m = \frac{1}{N_o} \frac{\sin[(2N_1 + 1)\omega/2]}{\sin(\omega/2)} \bigg|_{\omega = m\omega_o}$$

and sometimes the ratio of sine functions is called the *aliased sinc*.

There are two important observations about the DTFS, the first of which is something we discussed previously:

Remark 1 Since both $x[n]$ and $e^{-jm\omega_o n}$, for any m, are periodic sequences in *n* with period N_o, we can compute

$$X_m = \frac{1}{N_o} \sum_{n=0}^{N_o-1} x[n]\, e^{-jm\omega_o n}$$

by summing over any N_o consecutive values of the index *n*, not necessarily the values from zero to $N_o - 1$. This is often written as

$$X_m = \frac{1}{N_o} \sum_{n=<N_o>} x[n]\, e^{-jm\omega_o n}$$

where the special angle brackets denote an index range of " N_o consecutive values."

Remark 2 The DTFS coefficients $X_0, \ldots, X_{N_o-1}$ can be extended in either direction to form a sequence that repeats according to

$$X_{m+N_o} = X_m \,, \quad \text{for all } m$$

This follows from the calculation

$$\begin{aligned} X_{m+N_o} &= \frac{1}{N_o} \sum_{n=0}^{N_o-1} x[n]\, e^{-j(m+N_o)\omega_o n} \\ &= \frac{1}{N_o} \sum_{n=0}^{N_o-1} x[n]\, e^{-jm\frac{2\pi}{N_o}n} e^{-j2\pi n} \\ &= X_m \end{aligned}$$

A consequence is that, for any integer *k*,

$$X_k e^{jk\omega_o n} = X_{N_o+k} e^{j(N_o+k)\omega_o n}$$

Thus we can write

$$x[n] = \sum_{m=0}^{N_o-1} X_m e^{jm\omega_o n} = \sum_{m=<N_o>} X_m e^{jm\omega_o n}$$

where again the angle-bracket notation indicates a summation over N_o consecutive values of the index *m*.

Example The signal $x[n] = \sin(2\pi n/3)$ is periodic with fundamental period $N_o = 3$. We can calculate the DTFS coefficients as follows, though some details are omitted:

$$X_0 = \frac{1}{3} \sum_{n=0}^{2} \sin(2\pi n/3) = 0$$

$$X_1 = \frac{1}{3} \sum_{n=0}^{2} \sin(2\pi n/3)\, e^{-j\frac{2\pi}{3}n} = \frac{1}{2j}$$

$$X_2 = \frac{1}{3} \sum_{n=0}^{2} \sin(2\pi n/3)\, e^{-j2\frac{2\pi}{3}n} = \frac{-1}{2j}$$

However, there is a shortcut available. Simply write

$$\sin(2\pi n/3) = \frac{1}{2j} e^{j\frac{2\pi}{3}n} - \frac{1}{2j} e^{-j\frac{2\pi}{3}n}$$

and compare this to the expression

$$x[n] = \sum_{m=<3>} X_m e^{jm\frac{2\pi}{3}n}$$

Choosing the index range $<3> = -1, 0, 1$ we see that

$$X_{-1} = \frac{-1}{2j},\ X_0 = 0,\ X_1 = \frac{1}{2j}$$

These two results can be reconciled by noting that $X_2 = X_{-1+3} = X_{-1}$. Indeed, the discrete-time phasor corresponding to the coefficient X_2 is $e^{j2\frac{2\pi}{3}n}$ and this is identical to the phasor $e^{-j\frac{2\pi}{3}n}$ corresponding to X_{-1}, as is easily verified.

The first applet in the demonstration linked below permits you to explore the DTFS for signals with period 5. Select the "Input *x*[*n*]" option and set the speed to "slow." Then you can sketch in a signal and "play" the DTFS frequency components. Other options permit you to enter the coefficients in the DTFS or enter the magnitude and phase spectra, and these will be useful in Section 9.2. (You will need to use MSIE 5.5+ with the MathPlayer plugin to use this link. See the main demonstrations page for other versions of the applet.)

Discrete-Time Fourier Series

9.2 Spectra of DT Signals

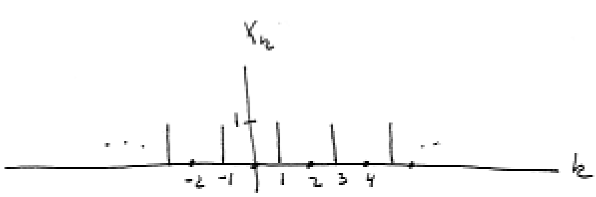
For a real, periodic, DT signal $x[n]$, the frequency content of the signal is revealed by the coefficients in the DTFS expression

$$x[n] = \sum_{m=<N_o>} X_m\, e^{jm\omega_o n}$$

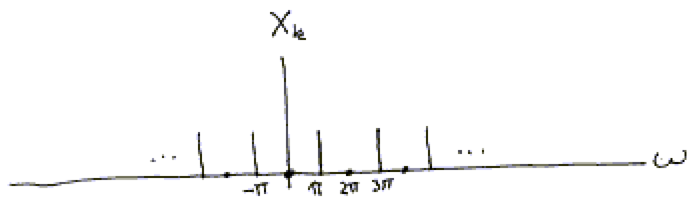
The following graphical displays of these coefficients define various *spectra* of $x[n]$.

The *magnitude spectrum* of $x[n]$ is a line plot of $|X_m|$ vs the index *m*, or vs $m\omega_o$ on a frequency axis. The *phase spectrum* of $x[n]$ is a similar plot of $\angle X_m$, usually on an angular range from $-\pi$ to π. Finally, when the DTFS coefficients are all real, the *amplitude spectrum* of $x[n]$ is simply a plot of the coefficients X_m.

Example In Section 9.1 we computed the DTFS coefficients of $x[n] = (-1)^n$ as $X_0 = 0$, $X_1 = 1$, and $X_{k+2} = X_k$, for other values of *k*. In this case the amplitude spectrum of the signal is simply



or, in terms of a frequency axis, since $\omega_o = \pi$,

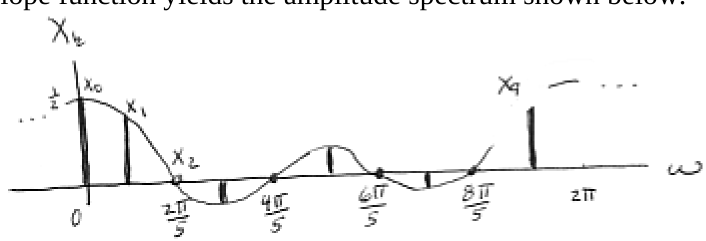


Since π corresponds to the highest frequency in discrete time, we note the obvious fact that $x[n]$ is a high-frequency signal. Finally, for this simple case, the magnitude spectrum is identical to the amplitude spectrum, and the phase spectrum is zero.

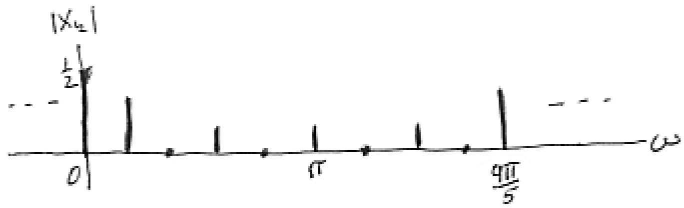
Example The second example in Section 9.1, a periodic train of width $2N_1 + 1$ clumps of unit-height lollypops, is considerably more complicated, though again the DTFS coefficients are real. Choosing $N_1 = 2,\ N_o = 10$, the coefficients are given as an evaluation of an *aliased sinc* envelop by

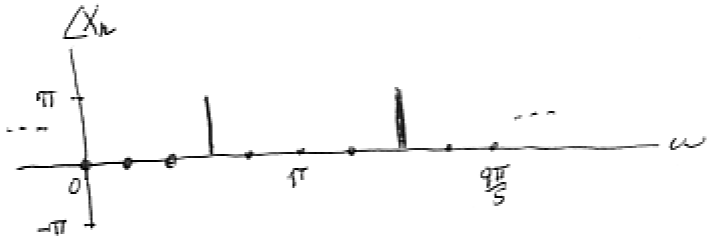
$$X_m = \frac{1}{10}\,\frac{\sin(5\omega/2)}{\sin(\omega/2)}\bigg|_{\omega = m\pi/5}$$

The envelope function is zero when $5\omega/2 = k\pi$, for nonzero integer *k*, that is, for $\omega = 2k\pi/5$. Sketching this envelope function yields the amplitude spectrum shown below.



Correspondingly, the magnitude and phase spectra are shown below.





While there is some high-frequency content in $x[n]$, in particular the component at frequency π, there is more low-frequency content as indicated by the components near the frequencies zero and 2π. Finally, it should be noted that these spectra repeat outside the frequency ranges shown.

To explore the notion of spectra in more detail, consult the demonstration *Discrete-Time Fourier Series* linked at the end of Section 9.1.

9.3 Operations on Signals

Discrete-time, periodic signals are completely determined by the fundamental frequency, ω_o, or fundamental period, N_o, and any N_o consecutive Fourier coefficients, say, $X_0, X_1, ..., X_{N_o-1}$. Thus the signal is described in terms of its frequency content. This raises the possibility of interpreting various time-domain operations on signals as operations on the frequency-domain description. However, rather than give a lengthy treatment of this issue, we will simply discuss a few examples.

Example 1 Given a signal

$$x[n] = \sum_{k=<N_o>} X_k \, e^{jk\omega_o n}$$

suppose a new signal is obtained by the index shift

$$\hat{x}[n] = x[n - n_o]$$

where n_o is a fixed integer. Clearly a shift does not change periodicity, or the fundamental period, or fundamental frequency. Therefore we can compute the Fourier coefficients for $\hat{x}[n]$ from the standard formula:

$$\hat{X}_k = \frac{1}{N_o} \sum_{n=<N_o>} \hat{x}[n] e^{-jk\omega_o n} = \frac{1}{N_o} \sum_{n=<N_o>} x[n - n_o] e^{-jk\omega_o n}$$

Changing the summation index from n to $m = n - n_o$,

$$\hat{X}_k = \frac{1}{N_o} \sum_{m=<N_o>} x[m] e^{-jk\omega_o (m+n_o)} = e^{-jk\omega_o n_o} \frac{1}{N_o} \sum_{n=<N_o>} x[n] e^{-jk\omega_o m}$$

$$= e^{-jk\omega_o n_o} X_k$$

Notice that the magnitude spectrum of the signal is unchanged by time shift, since, regardless of the integer value of *k*,

$$\left|\hat{X}_k\right| = \left|e^{-jk\omega_o N_o}\right| \left|X_k\right| = \left|X_k\right|$$

In simple cases, such as time-index shift, it is possible to ascertain the effect of the operation on the Fourier coefficients by inspection of the representation. Indeed, with $x[n]$ as given above, it is clear that

$$\hat{x}[n] = x[n - n_o] = \sum_{k=<N_o>} X_k \, e^{jk\omega_o (n-n_o)} = \sum_{k=<N_o>} e^{-jk\omega_o n_o} X_k \, e^{jk\omega_o n}$$

and we simply recognize the form of the expression and the corresponding Fourier coefficients $\hat{X}_k$.

Example 2 Suppose $\hat{x}[n] = x[-n]$. Again, the fundamental frequency does not change, and the Fourier coefficients for $\hat{x}[n]$ are given by

$$\hat{X}_k = \frac{1}{N_o} \sum_{n=<N_o>} \hat{x}[n] e^{-jk\omega_o n} = \frac{1}{N_o} \sum_{n=<N_o>} x[-n] e^{-jk\omega_o n}$$

Changing the summation index to $m = -n$ gives

$$\hat{X}_k = \frac{1}{N_o} \sum_{m=<N_o>} x[m]e^{-jk\omega_o(-m)} = \frac{1}{N_o} \sum_{m=<N_o>} x[m]e^{-j(-k)\omega_o n}$$

$$= X_{-k}$$

This conclusion also could be reached by inspection of the representation.

Further discussion of operations on discrete-time signals can be found in the demonstration DTFS Properties

9.4 DT LTI Frequency Response and Filtering

For a stable DT LTI system with periodic input signal, the DTFS and the eigenfunction property provide a way to compute and interpret the response. If the unit-pulse response of the system is $h[n]$, we define the *frequency response function* of the system as

$$H(\omega) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n}$$

This is a slight change in notation from Section 5.5 in that we show frequency as a variable. Note also that

$$H(\omega + 2\pi) = \sum_{n=-\infty}^{\infty} h[n]e^{-j(\omega+2\pi)n}$$

$$= H(\omega)$$

so in discrete time the frequency response function repeats every 2π radians in frequency.

If the input signal $x[n]$ is periodic, with fundamental period N_o and corresponding fundamental frequency ω_o, we can write

$$x[n] = \sum_{k=<N_o>} X_k\, e^{jk\omega_o n}$$

and the eigenfunction property gives

$$y[n] = \sum_{k=<N_o>} H(k\omega_o) X_k\, e^{jk\omega_o n}$$

Thus the frequency response function of the system describes the effect of the system on various frequency components of the input signal. To display this effect, plots of $|H(\omega)|$ and $\angle H(\omega)$ vs ω can be given. Since these functions repeat with period 2π, we often show the plots for only this range, for example, for $-\pi < \omega \le \pi$.

Example Suppose an LTI system has the unit pulse response

$$h[n] = \tfrac{1}{2}\delta[n] + \tfrac{1}{2}\delta[n-1]$$

Obviously the system is stable, and the frequency response function is

$$H(\omega) = \frac{1}{2} + \frac{1}{2}e^{-j\omega} = e^{-j\omega/2}\cos(\omega/2)$$

The response of the system to an input of frequency ω_o, for example,

$$x[n] = \cos(\omega_o n) = \text{Re}\left\{e^{j\omega_o n}\right\}$$

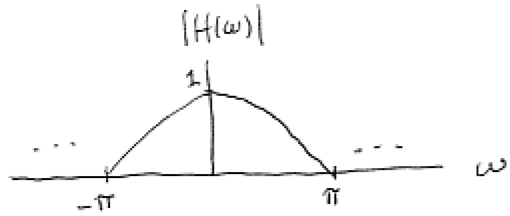
is given by a now-standard calculation using the eigenfunction property:

$$y[n] = \text{Re}\left\{H(\omega_o)e^{j\omega_o n}\right\} = |H(\omega_o)|\cos(\omega_o n + \angle H(\omega_o))$$

In this case,

$$|H(\omega)| = \cos(\omega/2)\,, \quad -\pi < \omega \le \pi$$

and from the plot below we see that the system is a low-pass filter.



Example Suppose an LTI system is described by the difference equation

$$y[n] + ay[n-1] = bx[n]$$

where, to guarantee stability, we assume $|a| < 1$. Then

$$h[n] = (-a)^n bu[n]$$

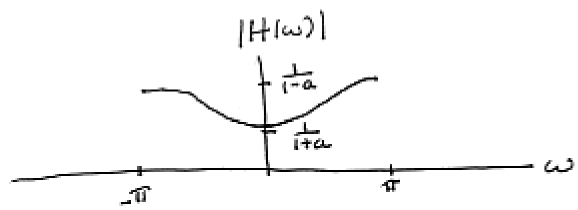
and the frequency response function is

$$H(\omega) = \sum_{n=-\infty}^{\infty} (-a)^n bu[n]\, e^{-j\omega n} = b\sum_{n=0}^{\infty} (-a)^n\ e^{-j\omega n}$$
$$= \frac{b}{1+ae^{-j\omega}}$$

In this case,

$$|H(\omega)| = \frac{|b|}{|1+ae^{-j\omega}|} = \frac{|b|}{\sqrt{(1+ae^{-j\omega})(1+ae^{j\omega})}} = \frac{|b|}{\sqrt{1+2a\cos(\omega)+a^2}}$$

If $0 < a < 1$ and $b = 1$, then the magnitude of the frequency response is shown below, and this system is a high-pass filter.

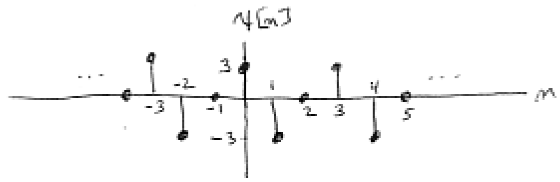


Exercises

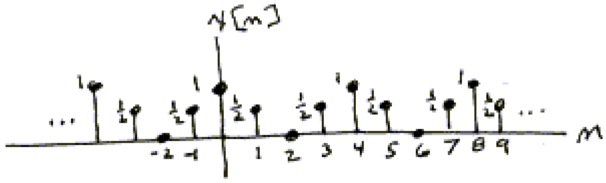
1. Compute the discrete-time Fourier series coefficients for the signals below and sketch the magnitude and phase spectra..

(a) $x[n] = 1 + \cos(\pi n/3)$

(b)



(c)



(d) $x[n] = \sum_{k=-\infty}^{\infty} \delta(n-4k-1)$

2. For the sets of DTFS coefficients given below, determine the corresponding real, periodic signal $x[n]$.

(a) $X_k = \begin{cases} 1/2, & k \text{ even} \\ -1/2, & k \text{ odd} \end{cases}$, $\omega_o = \pi$

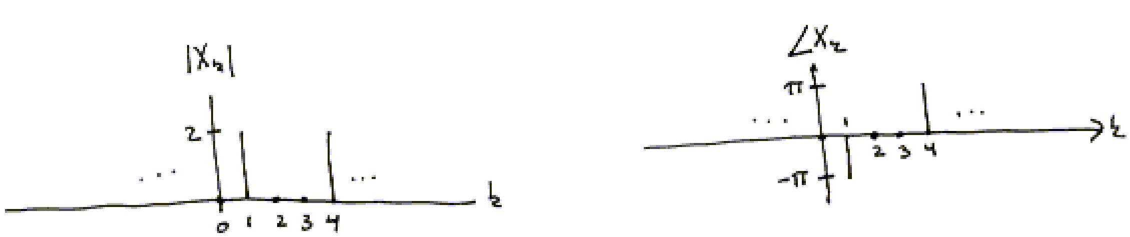
(b) $X_k = 1/2, \quad \textit{for all } k$, $\omega_o = \pi$

(c) $X_0 = -1, X_1 = 0, X_2 = 1, X_3 = -2, X_4 = 1, X_5 = 0, X_{k+6} = X_k, \ \omega_o = \pi/3$

3. Suppose $x[n]$ is periodic with *even* fundamental period N_o and DTFS coefficients X_k. If $x[n]$ also satisfies $x[n] = -x[n + N_o/2]$, for all *n*, show that $X_k = 0$ if *k* is even.

4. Given the fundamental period N_o and the magnitude and phase spectra as shown for a real, discrete-time signal, what is the signal?

(a) $N_o = 5$



5. If $x[n]$ has fundamental period N_o, an even integer, and discrete-time Fourier series coefficients X_k, what are the Fourier series coefficients for

(a) $\hat{x}[n] = x[n + N_o / 2]$

(b) $\hat{x}[n] = (-1)^n x[n]$ (Assume that $\hat{N}_o = N_o$ and give an example to show why this assumption is needed.)

6. For the LTI systems specified below, sketch the magnitude of the frequency response function and determine if the system is a low-pass or high-pass filter.

(a) $h[n] = \frac{1}{2}\delta[n] - \frac{1}{2}\delta[n-1]$

(b) $h[n] = \delta[n] - (1/2)^n u[n]$

(c) $h[n] = (1/2)^n u[n]$

Notes for Signals and Systems

10.1 Introduction to the CT Fourier Transform

If $x(t)$ is T_o-periodic, we can write the Fourier series description

$$x(t) = \sum_{k=-\infty}^{\infty} X_k \, e^{jk\omega_o t} \qquad (10.1)$$

where $\omega_o = 2\pi / T_o$ and

$$X_k = \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} x(t) e^{-jk\omega_o t} dt \,, \quad k = 0, \pm 1, \pm 2, \ldots \qquad (10.2)$$

Consider what happens as we let T_o grow without bound. In a sense, $x(t)$ approaches an aperiodic signal, and in any given frequency range, say $-1 \le \omega \le 1$, there are more and more frequency components of $x(t)$ since ω_o becomes smaller and smaller. Therefore, from a frequency content viewpoint, perhaps it is not surprising that a truely aperiodic signal typically contains frequency components at *all* frequencies, not just at integer multiples of a fundamental frequency.

Analysis of a limit process by which a periodic signal approaches an aperiodic signal, as $T_o \to \infty$, is difficult to undertake. Therefore we skip mathematical details and simply provide a motivational argument leading to a description of the frequency content of aperiodic signals.

Define, for all ω, the complex-valued function $X(\omega)$ by

$$X(\omega) = \int_{-T_o/2}^{T_o/2} x(t) e^{-j\omega t} dt \qquad (10.3)$$

The Fourier series coefficients (10.2) for $x(t)$ can be written as evaluations of this "envelope" function,

$$X_k = \frac{1}{T_o} X(k\omega_o) = \frac{1}{2\pi} X(k\omega_o)\omega_o$$

so we can write

$$x(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(k\omega_o) \, e^{jk\omega_o t} \omega_o$$

Letting T_o grow without bound, that is, letting ω_o shrink toward 0, we can view ω_o as a differential $d\omega$ and consider that $k\omega_o$ takes on the character of a real variable, ω. That is, the difference $(k+1)\omega_o - k\omega_o = \omega_o$ shrinks toward 0 so that any given real number can be approximated by $k\omega_o$, for suitable *k*. Then the sum transfigures to an integral, yielding

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \qquad (10.4)$$

In this expression, since we let $T_o \to \infty$, (10.3) becomes

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \qquad (10.5)$$

The function $X(\omega)$ is defined as the *Fourier transform* of $x(t)$, and in a very useful sense it describes the frequency content of the aperiodic signal. Indeed, mathematically, $x(t)$ is given by the *inverse Fourier transform* expression (10.4), which is analogous to the expression of a T_o-periodic signal in terms of its Fourier series.

To perform a sanity check on these claims of a transformation and an inverse transformation, suppose that given $x(t)$ we compute

$$X(\omega) = \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau$$

where we have used a different name for the integration variable in order to avoid a looming notational collision. Substituting this into the right side of (10.4) gives

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau \, e^{j\omega t} d\omega$$

Interchanging the order of integration gives

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} e^{j\omega(t-\tau)} d\omega \, d\tau$$

In this expression we recognize that

$$\int_{-\infty}^{\infty} e^{j\omega(t-\tau)} d\omega = 2\pi \, \delta(t-\tau)$$

from Special Property 2 in Section 2.2. Then the integration with respect to τ is evaluated by the sifting property to give

$$\int_{-\infty}^{\infty} x(\tau) \, 2\pi\delta(t-\tau) \, d\tau = x(t)$$

In other words, taking the Fourier transform and then the inverse transform indeed returns the original signal.

Immediate questions are: When is $X(\omega)$ a meaningful representation for $x(t)$? Are standard operations on $x(t)$ easy to interpret as operations on $X(\omega)$? Before addressing these, we work a few examples.

Example For $x(t) = e^{-3t} u(t)$,

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{0}^{\infty} e^{-3t} \, e^{-j\omega t} dt = \frac{-1}{3+j\omega} e^{-(3+j\omega)t} \Big|_{0}^{\infty}$$

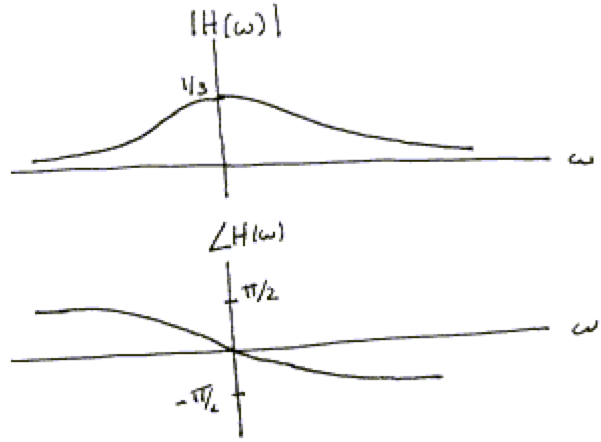
$$= \frac{1}{3+j\omega}$$

It is important to note that the evaluation at $t = \infty$ yields zero. For example, if we change the sign of the exponent in the signal, the Fourier transform would not exist.

In fact the Fourier transform $X(\omega)$ describes the frequency content of the signal $x(t)$, and we display this content using the following, familiar plots. The *magnitude spectrum* of a signal $x(t)$ is a plot of $|X(\omega)|$ vs ω, and the *phase spectrum* is a plot of $\angle X(\omega)$ vs ω. If $X(\omega)$ is a real-valued function, sometimes we plot the *amplitude spectrum* of the signal as $X(\omega)$ vs ω. The magnitude spectrum or amplitude spectrum of a signal display the frequency content of the signal. For the example above,

$$|X(\omega)| = \frac{1}{\sqrt{9+\omega^2}}, \quad \angle X(\omega) = -\tan^{-1}(\omega/3)$$

and the magnitude and phase spectra are shown below.



Remark The symmetry properties of the magnitude and phase spectra of a real signal $x(t)$ are easy to justify in general from the fact that

$$X^*(\omega) = \left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right]^* = \int_{-\infty}^{\infty} [x(t)e^{-j\omega t}]^* dt = \int_{-\infty}^{\infty} x(t)e^{j\omega t}dt$$

$$= \int_{-\infty}^{\infty} x(t)e^{-j(-\omega)t}dt = X(-\omega)$$

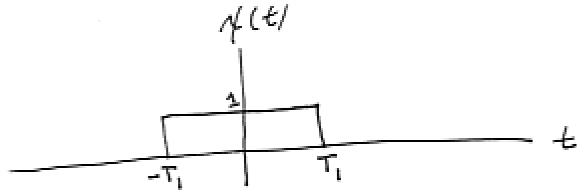
This gives $|X(-\omega)| = |X(\omega)|$ and $\angle X(-\omega) = -\angle X(\omega)$, and thus the magnitude spectrum is an even function of ω, while the phase spectrum is an odd function of ω.

Example A simple though somewhat extreme example is the unit impulse, $x(t) = \delta(t)$. The sifting property immediately gives

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = 1$$

In this case the magnitude (or amplitude) spectrum of the signal is a constant, indicating that the unit impulse is made up of equal amounts of all frequencies!

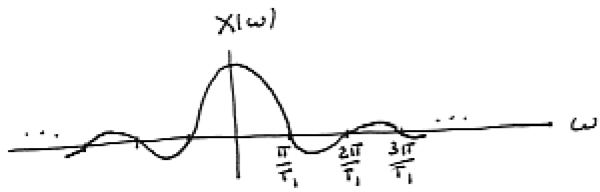
Example Consider the rectangular pulse signal shown below.



The Fourier transform computation involves a bit of work to put the answer in a nice form, but the calculations are not unfamiliar:

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-T_1}^{T_1} e^{-j\omega t}dt = \frac{-1}{j\omega} e^{-j\omega t}\Big|_{-T_1}^{T_1} \\ &= \frac{1}{j\omega}(e^{j\omega T_1} - e^{-j\omega T_1}) = 2T_1 \tfrac{1}{\omega T_1} \sin(\omega T_1) \\ &= 2T_1 \operatorname{sinc}(\omega T_1/\pi) \end{aligned}$$

Thus the amplitude spectrum of $x(t)$ is



As with Fourier series calculations, if the Fourier transform can be written as a real function, then it is important to express it in real form.

Convergence Issues It is difficult to explicitly characterize the class of signals for which the Fourier transform is well defined. Furthermore, the uniqueness of the Fourier transform for a given signal is an important issue – to each signal $x(t)$ there should correspond exactly one $X(\omega)$. (This neglects trivial changes in the signal, or the transform, for example adjusting the value of $x(t)$ at isolated values of *t*. Such a change does not effect the result of the integration leading to the transform.) There are various sufficient conditions that can be stated for a signal to have a unique Fourier transform, and we present only the best known of these.

Dirichlet Condition Suppose a signal $x(t)$ is such that

(a) $x(t)$ is absolutely integrable, that is, $\int_{-\infty}^{\infty} |x(t)|\,dt < \infty$,

(b) $x(t)$ has no more than a finite number of minima and maxima in any finite time interval, and

(c) $x(t)$ has no more than a finite number of discontinuities in any finite time interval, and these discontinuities are finite.

Then there exists a unique Fourier transform $X(\omega)$ corresponding to $x(t)$.

It is important to remember that this is a sufficient condition, and there are signals that do not satisfy the condition yet have a unique Fourier transform. For example, the unit impulse would be thought of as having an infinite discontinuity at $t = 0$, and yet the sifting property gives a Fourier transform. Indeed, we can check the calculation by applying the inverse transform. With $X(\omega) = 1$, we compute

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} d\omega = \delta(t)$$

where we have used again Special Property 2 from Section 2.2, that is,

$$\int_{-\infty}^{\infty} e^{j\omega t} d\omega = 2\pi \ \delta(t)$$

Example As an additional example, we can use the special property with the roles of *t* and ω interchanged to compute the Fourier transform of $x(t) = 1$:

$$X(\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} \ dt = \int_{-\infty}^{\infty} e^{j(-\omega)t} \ dt = 2\pi\delta(-\omega) = 2\pi\delta(\omega)$$

This indicates that all the frequency content in the signal is concentrated at zero frequency, a reasonable conclusion. A quick check of the inverse transform reassures:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(\omega) e^{j\omega t} d\omega$$
$$= 1$$

Indeed, one approach to computing a Fourier transform when the Dirichlet condition is not satisfied or generalized functions might be involved is to guess the transform and verify by use of the inverse transform formula.

10.2 Fourier Transform for Periodic Signals

If $x(t)$ is T_o-periodic, then it is clear that the Fourier transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

does not exist in the usual sense, because of the failure of the integral to converge. However, we can take an indirect approach and use notions of generalized functions to extend the Fourier transform to periodic signals in a way that captures the Fourier series expression in a fashion consistent with ordinary Fourier transforms. The Fourier series expression

$$x(t) = \sum_{k=-\infty}^{\infty} X_k \ e^{jk\omega_o t}$$

indicates that the key is to develop the Fourier transform of the complex signal

$$x(t) = e^{j\omega_o t}$$

One approach is to use the Special Property 2 in Section 2.2 again:

$$X(\omega) = \int_{-\infty}^{\infty} e^{j\omega_o t} \ e^{-j\omega t} dt = \int_{-\infty}^{\infty} e^{j(\omega_o - \omega)t} \ dt = 2\pi\delta(\omega_o - \omega)$$
$$= 2\pi\delta(\omega - \omega_o)$$

That this is reasonable is easily verified using the inverse Fourier transform, with evaluation by the sifting property.

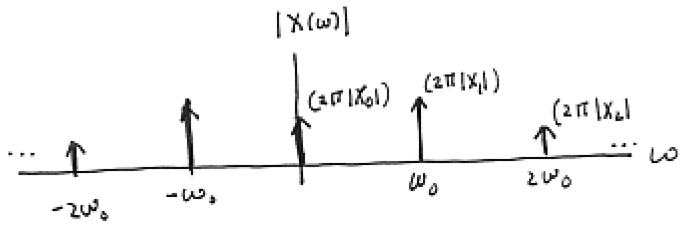
Following this calculation, we can compute the Fourier transform of a signal expressed by a Fourier series in a straightforward manner:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} X_k\ e^{jk\omega_o t}e^{-j\omega t}dt$$

$$= \sum_{k=-\infty}^{\infty} X_k \int_{-\infty}^{\infty} e^{jk\omega_o t}e^{-j\omega t}dt\ = \sum_{k=-\infty}^{\infty} 2\pi X_k\ \delta(\omega - k\omega_o)$$

In words, to compute the Fourier transform of a periodic signal, first compute the Fourier series coefficients, X_k , and then simply substitute this data into the $X(\omega)$ expression above.

Of course the Fourier transform expression is invertible by inspection, in the sense that the Fourier series for $x(t)$ can be written by inspection from $X(\omega)$. From another viewpoint, we essentially have built the Fourier series into the transform. We need to reformat the notions of spectra of $x(t)$ in consonance with this new framework, but that is easy. For any value of ω, at most one summand in the expression for $X(\omega)$ can be nonzero. Because the summands are non-overlapping in this sense, the magnitude of the sum is the sum of the magnitudes. Instead of a line plot, however, the magnitude spectrum (or amplitude spectrum, if every X_k is real) becomes a plot of impulse functions, occurring at integer multiples of the fundamental frequency, labeled with the impulse-area magnitudes. That is,



The phase spectrum computes in a similar fashion, since the angle of a sum of non-overlapping terms is the sum of the angles. Therefore the phase spectrum is interpreted as a line plot of the angles of the areas of impulses vs frequency. That is, the phase spectrum has exactly the same form as in the context of Fourier series representations.

Example A couple of special cases are interesting to note, based on the Fourier transform of a phasor. First, for

$$x(t) = \cos(\omega_o t) = \tfrac{1}{2}e^{j\omega_o t} + \tfrac{1}{2}e^{-j\omega_o t}$$

we have

$$X(\omega) = \pi\delta(\omega - \omega_o) + \pi\delta(\omega + \omega_o)$$

Second, for $x(t) = \sin(\omega_o t)$,

$$X(\omega) = -j\pi\delta(\omega - \omega_o) + j\pi\delta(\omega + \omega_o)$$

Example Other approaches to computing the Fourier transform of periodic signals can give “correct,” but difficult to interpret results. For the signal

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - k)$$

we directly compute

$$X(\omega) = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \delta(t-k)\, e^{-j\omega t} dt = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(t-k)\, e^{-j\omega t} dt$$

$$= \sum_{k=-\infty}^{\infty} e^{-j\omega k}$$

On the other hand, the recommended procedure is to compute the Fourier series data for $x(t)$, which easily yields $\omega_o = 2\pi$ and $X_k = 1$, for all *k*. Then by inspection we obtain the Fourier transform

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi\, \delta(\omega - k2\pi)$$

Needless to say, it is difficult to show by elementary means that these two expressions for $X(\omega)$ are the same. In any case, we always prefer the second.

10.3 Properties of the Fourier Transform

We now consider a variety of familiar operations on a signal $x(t)$, and interpret the effect of these operations on the corresponding Fourier transform $X(\omega)$. Of course, existence of the Fourier transform is assumed. Furthermore, we should verify that each operation considered yields a signal that has a Fourier transform. Often this is obvious, and will not be mentioned, but care is needed in a couple of cases. The proofs we offer of the various properties are close to being rigorous for ordinary signals, for example those satisfying the Dirichlet condition. For generalized functions, or signals such as periodic signals that have generalized-function transforms, further interpretation typically is needed.

Throughout we use the following notation for the Fourier transform and inverse Fourier transform, where *F* denotes a "Fourier transform operator:"

$$X(\omega) = F[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = F^{-1}[X(\omega)] = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Linearity If $F[x(t)] = X(\omega)$ and $F[z(t)] = Z(\omega)$, then for any constant *a*,

$$F[x(t) + a\, z(t)] = X(\omega) + a\, Z(\omega)$$

This property follows directly from the definition.

Time Shifting If $F[x(t)] = X(\omega)$, then for any constant t_o,

$$F[x(t - t_o)] = e^{-j\omega t_o} X(\omega)$$

The calculation verifying this is by now quite standard. Begin with

$$F[x(t - t_o)] = \int_{-\infty}^{\infty} x(t - t_o) e^{-j\omega t} dt$$

and change integration variable from *t* to $\tau = t - t_o$ to obtain

$$F[x(t-t_o)]=\int_{-\infty}^{\infty} x(\tau)e^{-j\omega(\tau+t_o)}dt = e^{-j\omega t_o}\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}dt$$
$$= e^{-j\omega t_o}X(\omega)$$

Time Scaling If $F[x(t)] = X(\omega)$, then for any constant $a \neq 0$,

$$F[x(at)] = \frac{1}{|a|}X(\frac{\omega}{a})$$

This is another familiar calculation, beginning with

$$F[x(at)]=\int_{-\infty}^{\infty} x(at)e^{-j\omega t}dt$$

though the cases of positive and negative *a* are conveniently handled separately. If $a > 0$, the variable change from *t* to $\tau = at$ yields

$$F[x(at)]=\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\frac{\tau}{a}}\,\frac{1}{a}\,d\tau = \frac{1}{a}\int_{-\infty}^{\infty} x(\tau)e^{-j\frac{\omega}{a}\tau}d\tau$$
$$=\frac{1}{a}X(\frac{\omega}{a})$$

If $a < 0$, then it is convenient to write $a = -|a|$ and use the variable change $\tau = -|a|t$ as follows:

$$F[x(at)]=\int_{\infty}^{-\infty} x(\tau)e^{-j\omega\frac{\tau}{-|a|}}\,\frac{1}{-|a|}\,d\tau = \frac{1}{|a|}\int_{-\infty}^{\infty} x(\tau)e^{-j\frac{\omega}{a}\tau}d\tau$$
$$=\frac{1}{|a|}X(\frac{\omega}{a})$$

Finally we note that both cases are covered by the claimed formula.

Example An interesting case is $a = -1$, and the scaling property yields

$$F[x(-t)] = X(-\omega)$$

As noted previously, $|X(-\omega)| = |X(\omega)|$, and therefore we notice the interesting fact that a signal and its time reversal have the same magnitude spectra!

Example When both a scale and a shift are involved, the safest approach is to work out the result from the basic definition. To illustrate, the Fourier transform of $x(3t-2)$ can be computed in terms of the transform of $x(t)$ via the variable change $\tau = 3t-2$ as shown in the following:

$$F[x(3t-2)]=\int_{-\infty}^{\infty} x(3t-2)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\frac{\tau+2}{3}}\,\frac{1}{3}\,d\tau$$
$$=\frac{1}{3}e^{-j\omega\frac{2}{3}}\int_{-\infty}^{\infty} x(\tau)e^{-j\frac{\omega}{3}\tau}d\tau$$
$$=\frac{1}{3}e^{-j\omega\frac{2}{3}}X(\frac{\omega}{3})$$

Differentiation If $F[x(t)] = X(\omega)$, and the time-derivative signal $\dot{x}(t)$ has a Fourier transform, then

$$F[\dot{x}(t)] = j\omega\, X(\omega)$$

To justify this property, we directly compute the transform, using integration-by-parts:

$$\begin{aligned} F[\dot{x}(t)] &= \int_{-\infty}^{\infty} \dot{x}(t) e^{-j\omega t}\, dt = x(t)e^{-j\omega t}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} x(t)(-j\omega)e^{-j\omega t} d\tau \\ &= j\omega\, X(\omega) \end{aligned}$$

To make this rigorous, we need to justify the fact that $x(t)$ approaches zero as $t \to \pm\infty$. When $x(t)$ satisfies the Dirichlet condition, the fact is clear, though in other cases it is less so.

Remark When we apply this property to signals that are not, strictly speaking, differentiable, generalized calculus must be used. For example, beginning with the easily-verified transform

$$F[x(t)] = F[e^{-t}u(t)] = \frac{1}{1+j\omega}$$

the differentiation property gives

$$F[\dot{x}(t)] = \frac{j\omega}{1+j\omega}$$

To check this, we first compute $\dot{x}(t)$,

$$\begin{aligned} \dot{x}(t) &= \tfrac{d}{dt}\left(e^{-t}u(t)\right) = -e^{-t}u(t) + e^{-t}\,\delta(t) \\ &= -e^{-t}u(t) + \delta(t) \end{aligned}$$

Then the Fourier transform is easy by linearity:

$$F[\dot{x}(t)] = \frac{-1}{1+j\omega} + 1 = \frac{j\omega}{1+j\omega}$$

Integration If $F[x(t)] = X(\omega)$, then

$$F\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right] = \tfrac{1}{j\omega} X(\omega) + \pi X(0)\delta(\omega)$$

This property is difficult to derive, but we can observe that the running integral is the inverse of differentiation, except for an undetermined additive constant. Thus the first term is expected, and the second term accounts for the constant.

Example The relationships

$$F[\delta(t)] = 1, \qquad u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$

and the integration property give the Fourier transform of the unit-step function as

$$F[u(t)] = \tfrac{1}{j\omega} + \pi\delta(\omega)$$

We can check this with the differentiation property (using generalized differentiation):

$$\begin{aligned} F[\delta(t)] = F[\dot{u}(t)] &= j\omega\left[\tfrac{1}{j\omega} + \pi\delta(\omega)\right] = 1 + j\pi\omega\,\delta(\omega) \\ &= 1 \end{aligned}$$

though the rule for multiplying an impulse with an ordinary function must be invoked.

Example The properties we are discussing sometimes can be used in clever ways to compute Fourier transforms based on a small table of known transforms. However, sometimes the answer can appear in a form where interpretation is required to simplify the result. To illustrate, consider the simple rectangular pulse,

$$x(t) = u(t+1) - u(t-1)$$

Using linearity and time-shift properties, we can immediately write

$$\begin{aligned} X(\omega) &= e^{j\omega}\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] - e^{-j\omega}\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] \\ &= \frac{1}{j\omega}\left[e^{j\omega} - e^{-j\omega}\right] + \pi\left[e^{j\omega} - e^{-j\omega}\right]\delta(\omega) \\ &= \frac{2\sin(\omega)}{\omega} \\ &= 2\,\mathrm{sinc}(\omega/\pi) \end{aligned}$$

This agrees with our earlier conclusion, though, again, application of the rule for multiplying and impulse function by an ordinary function is involved.

The effect of various operations on a signal and on the magnitude and phase spectra of the signal can be explored using the Web demonstration below.

CTFT Properties

10.4 Convolution Property and Frequency Response of LTI Systems

Perhaps the most important property of the Fourier transform is that convolution in the time domain becomes multiplication of transforms. This means that for many purposes it is convenient to view LTI systems in the Fourier (frequency) domain.

Convolution If $x(t)$ and $h(t)$ have Fourier transforms $X(\omega)$ and $H(\omega)$, then

$$F[(x*h)(t)] = X(\omega)H(\omega)$$

A direct calculation involving a change in order of integration can be used to establish this property:

$$\begin{aligned} F[(x*h)(t)] &= \int_{-\infty}^{\infty} (x*h)(t)e^{-j\omega t}\,dt \\ &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau\right]e^{-j\omega t}\,dt \\ &= \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau)\,e^{-j\omega t}\,dt\right]d\tau \end{aligned}$$

We can change the integration variable in the inner integration from t to $\sigma = t - \tau$ to obtain

$$F[(x*h)(t)] = \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} h(\sigma)\,e^{-j\omega(\sigma+\tau)}\,d\sigma\right]d\tau$$

and factoring the exponential in τ out of the inner integration gives

$$F[(x*h)(t)] = \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau} H(\omega)\, d\tau = X(\omega)H(\omega)$$

Remark We did not check that the convolution of Fourier transformable signals yields a Fourier transformable signal. This in fact is true for signals that satisfy the Dirichlet condition, but difficulties can arise when some of our signals with generalized-function transforms are involved. For example, the transforms of the signals $x(t) = 1$ and $h(t) = u(t)$ are

$$X(\omega) = 2\pi\delta(\omega)\,, \qquad H(\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

The convolution $(x*h)(t)$ is undefined in this case. Fortunately the product of the transforms indicates that something is amiss in that the square of an impulse function at $\omega = 0$ appears, and also an impulse multiplied by a function that is dramatically discontinuous at $\omega = 0$. But such a clear indication is not always provided.

Example For a stable LTI system, the eigenfunction property states that the response to $x(t) = e^{j\omega_o t}$ is $y(t) = H(\omega_o)\, e^{j\omega_o t}$, where

$$H(\omega_o) = \int_{-\infty}^{\infty} h(t) e^{-j\omega_o t} dt$$

In terms of Fourier transforms, the convolution property implies that the response to $X(\omega) = 2\pi\delta(\omega - \omega_o)$ is

$$Y(\omega) = H(\omega) 2\pi\delta(\omega - \omega_o) = H(\omega_o) 2\pi\delta(\omega - \omega_o)$$

where $H(\omega)$ is the Fourier transform of the unit-impulse response $h(t)$. This is simply the eigenfunction property in terms of Fourier transforms – the output transform is a constant (typically complex) multiple of the input transform.

This example is easily extended to represent the response of a stable, LTI system to a periodic input signal. We view the periodic input signal in terms of its Fourier series,

$$x(t) = \sum_{k=-\infty}^{\infty} X_k\, e^{jk\omega_o t}$$

Then we immediately have

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi\, X_k \delta(\omega - k\omega_o)$$

and

$$Y(\omega) = H(\omega) X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi\, X_k H(\omega) \delta(\omega - k\omega_o)$$

$$= \sum_{k=-\infty}^{\infty} 2\pi\, X_k H(k\omega_o) \delta(\omega - k\omega_o)$$

The inverse Fourier transform gives

$$y(t) = \sum_{k=-\infty}^{\infty} X_k\, H(k\omega_o) e^{jk\omega_o t}$$

which typically is the Fourier series representation of the output signal.

If an LTI system is stable, then the frequency response function $H(\omega)$, which we now recognize as the Fourier transform of the unit-impulse response, is well defined, and we can view the input-output behavior of the system in terms of

$$Y(\omega) = H(\omega)X(\omega)$$

Then the magnitude spectrum of the output signal is related to the magnitude spectrum of the input signal by

$$|Y(\omega)| = |H(\omega)||X(\omega)|$$

Thus the magnitude of the frequency response function can be viewed as a frequency-dependent gain of the system.

Example An LTI system is said to exhibit *distortionless transmission* if the output signal is simply an amplitude scaled (positively) and time-delayed version of the input signal. That is, there are positive constants a and t_o such that for any input $x(t)$ the output signal is $y(t) = a\, x(t - t_o)$. That is, assuming the input signal has a Fourier transform, $Y(\omega) = a\, e^{-j\omega t_o} X(\omega)$, and thus we see that for distortionless transmission the frequency response function has the form

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = a\, e^{-j\omega t_o}$$

Often this is stated as the frequency response function must have "flat magnitude" and phase that is a linear function of frequency, at least for the frequency range of interest.

Example An *ideal filter* should transmit without distortion all frequencies in the specified frequency range, and remove all frequencies outside this range. For example,
and ideal low-pass filter should have the frequency response function

$$H(\omega) = \begin{cases} e^{-j\omega t_o}, & |\omega| \le \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

In this expression, $\omega_c > 0$ is the *cutoff frequency*, and for convenience we have set the constant gain to unity. To interpret this filter in the time domain, we can compute the impulse response via the inverse Fourier transform:

$$h(t) = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)\, e^{j\omega t}\, d\omega = \tfrac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega(t - t_o)}\, d\omega$$

$$= \tfrac{\omega_c}{\pi} \operatorname{sinc}[\omega_c (t - t_o)/\pi]$$

A key point is that no matter how large we permit the delay time t_o to be, this unit-impulse response is not right sided, and thus the ideal low-pass filter is not a causal system. Despite this drawback, the ideal low-pass filter remains a useful concept.

Example The LTI system described by

$$\dot{y}(t) + \omega_c y(t) = \omega_c x(t)$$

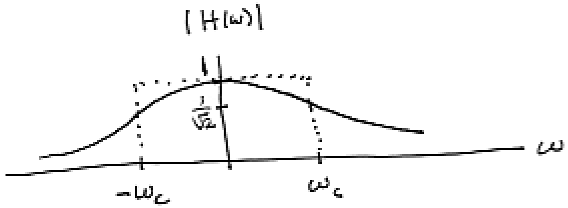
with $\omega_c > 0$, is a low-pass filter that can be implemented by an R-C circuit. Simple, familiar calculations give

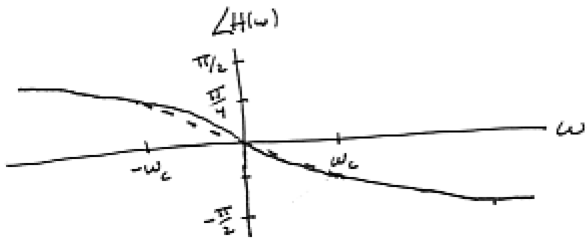
$$H(\omega) = \frac{\omega_c}{\omega_c + j\omega}$$

Expressing this frequency response function in polar form, as

$$H(\omega) = \frac{\omega_c}{\sqrt{\omega_c^2 + \omega^2}}\, e^{-j\tan^{-1}(\omega/\omega_c)}$$

the characteristics of this filter can be compared to the ideal low-pass filter via the magnitude and phase plots shown below. (For the ideal filter, we choose $t_o = \pi/(4\omega_c)$ to obtain an angle of $-\pi/4$ at $\omega = \omega_c$, for illustration.)





Clearly the first-order differential equation is a rather poor approximation to an ideal filter, but higher-order differential equations can perform closer to the ideal.

10.5 Additional Fourier Transform Properties

Frequency-Domain Convolution If $x(t)$ and $z(t)$ have Fourier transforms $X(\omega)$ and $Z(\omega)$, then

$$F[x(t)z(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\xi)Z(\omega-\xi)\, d\xi$$

That is, the Fourier transform of a product of signals is the convolution of the transforms: $\frac{1}{2\pi}(X * Z)(\omega)$.

To prove this property, we directly compute the inverse transform of $\frac{1}{2\pi}(X * Z)(\omega)$:

$$F^{-1}[\tfrac{1}{2\pi}(X * Z)(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\xi)Z(\omega-\xi)\, d\xi\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\xi)\, \frac{1}{2\pi}\int_{-\infty}^{\infty} Z(\omega-\xi)\, e^{j\omega t}\, d\omega\, d\xi$$

Changing the variable of integration in the inner integral from ω to $\eta = \omega - \xi$ gives

$$F^{-1}[\tfrac{1}{2\pi}(X*Z)(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\xi)\,\frac{1}{2\pi}\int_{-\infty}^{\infty} Z(\eta)\,e^{j(\eta+\xi)t}\,d\eta\,d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\xi)e^{j\xi t}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} Z(\eta)\,e^{j\eta t}\,d\eta\,d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\xi)e^{j\xi t}\,z(t)\,d\xi$$

$$= x(t)\,z(t)$$

Example The most important application of this property is the so-called *modulation* or *frequency shifting* property. If $z(t) = e^{j\omega_o t}$, then the sifting property of impulses gives

$$F[e^{j\omega_o t}x(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\xi)\,2\pi\,\delta(\omega-\xi-\omega_o)\,d\xi$$

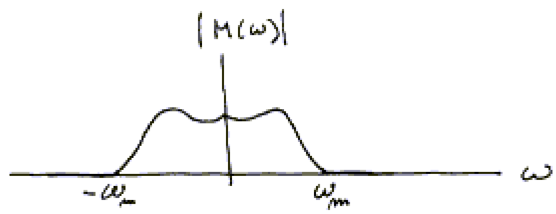
$$= X(\omega-\omega_o)$$

We can illustrate this property by considering the structure of *AM* radio signals.

Amplitude Modulated Signals An amplitude modulated (*AM*) signal of the most basic type has the form

$$x(t) = [1 + k\,m(t)]\cos(\omega_c t)$$

where $\cos(\omega_c t)$ is called the *carrier signal*, $m(t)$ is called the *message signal*, and the constant k is called the *modulation index*. We assume that the modulation index is such that $1 + k\,m(t) \ge 0$, for all t. We also assume that the message signal is *bandlimited* by ω_m, where $\omega_m \ll \omega_c$. That is, the Fourier transform $M(\omega)$ of the message signal is zero outside the frequency range $-\omega_m \le \omega \le \omega_m$. These are standard situations in practice, and we will represent the magnitude spectrum of the message signal as shown below.



Under these assumptions, $x(t)$ has the form of a high-frequency (rapidly oscillating) sinusoid with a relatively slowly-varying amplitude envelope. We could sketch a typical case in the time domain, but it would be rather uninformative as to the special properties of *AM* signals that make them so useful in communications. To reveal these properties, we turn to the frequency domain via the Fourier transform.

Using the Fourier transform

$$F[\cos(\omega_c t)] = \pi\,\delta(\omega-\omega_c) + \pi\,\delta(\omega+\omega_c)$$

and the frequency-domain convolution

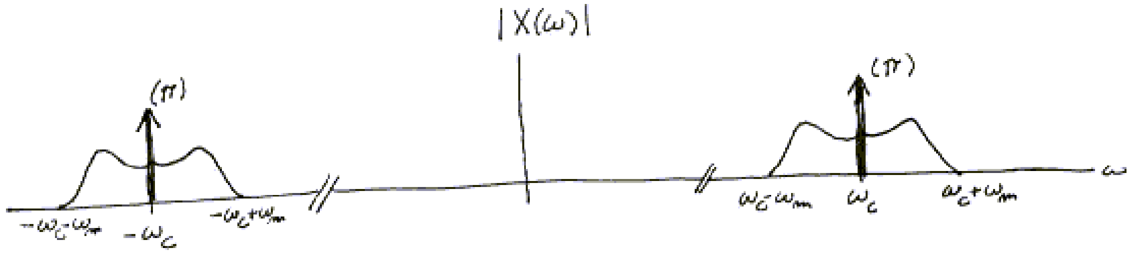
$$F[m(t)\cos(\omega_c t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} M(\xi)[\pi\,\delta(\omega-\xi-\omega_c)+\pi\,\delta(\omega-\xi+\omega_c)]\,d\xi$$

$$= \tfrac{1}{2}M(\omega-\omega_c)+\tfrac{1}{2}M(\omega+\omega_c)$$

we conclude that the transform of the *AM* signal is

$$X(\omega) = \pi\,\delta(\omega-\omega_c)+\pi\,\delta(\omega+\omega_c)+\tfrac{k}{2}M(\omega-\omega_c)+\tfrac{k}{2}M(\omega+\omega_c)$$

To see the properties of an *AM* signal, we consider the magnitude spectrum, $|X(\omega)|$. Typically it is difficult to compute the magnitude of a sum, but because of the assumptions on the highest message frequency and carrier frequency, at most one term in the sum is nonzero at every frequency, except at the carrier frequency. (At the carrier frequency, we have an impulse and an ordinary value, and we can display this situation graphically in the obvious way.) Because of this special structure of the terms, the magnitude of this particular sum essentially is the sum of the magnitudes, and we obtain



From this plot is clear that AM modulation is used to shift the spectral content of a message to a frequency range reserved for a particular transmitter. By assigning different carrier frequencies to different transmitters, with a separation of at least $2\omega_m$ in the different carriers, the messages are kept distinct.

Parseval's Theorem If $x(t)$ is a real energy signal with Fourier transform $X(\omega)$, then the total energy of the signal is given by

$$\int_{-\infty}^{\infty} x^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\,d\omega$$

To establish this result, we substitute the inverse Fourier transform expression for one of the $x(t)$'s in the time-domain energy expression to obtain

$$\int_{-\infty}^{\infty} x^2(t)\,dt = \int_{-\infty}^{\infty} x(t)\ \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega\,dt\ = \int_{-\infty}^{\infty} X(\omega)\ \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)e^{j\omega t}\,dt\,d\omega$$

The inner integral in this expression can be recognized as the conjugate of the Fourier transform of $x(t)$, $X^*(\omega)$, and therefore

$$\int_{-\infty}^{\infty} x^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,X^*(\omega)\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\,d\omega$$

The importance of Parseval's theorem is that energy can be associated with frequency content. For example,

$$\int_{-3}^{3} |X(\omega)|^2 d\omega$$

is the portion of the energy of $x(t)$ that resides in the frequency band $-3 \le \omega \le 3$.

Duality Property If $x(t)$ has Fourier transform $X(\omega)$, then

$$F[X(t)] = 2\pi\, x(-\omega)$$

This property can be recognized from an inspection of the Fourier and inverse Fouier transform expression. However, we will be pedantic and list out the appropriate sequence of variable changes. Beginning with

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega$$

change the variable of integration from ω to $\hat{t}$ and then replace the variable t by $-\hat{\omega}$. This gives

$$x(-\hat{\omega}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\hat{t}) e^{-j\hat{\omega}\hat{t}}\, d\hat{t}$$

Now change variables from $\hat{\omega}$ to ω, and $\hat{t}$ to t (erase the *hats*) to obtain

$$x(-\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(t) e^{-j\omega t}\, dt = \frac{1}{2\pi} F[X(t)]$$

One use of the duality property is in reading tables of Fourier transforms backwards to generate additional Fourier transforms!

Examples Since $F[\delta(t)] = 1$, the duality property gives $F[1] = 2\pi\, \delta(-\omega) = 2\pi\, \delta(\omega)$. A more usual example is that since

$$F\left[e^{-t} u(t)\right] = \frac{1}{1 + j\omega}$$

the duality property gives

$$F\left[\frac{1}{1 + jt}\right] = 2\pi\, e^{\omega} u(-\omega)$$

Thus we see that since many Fourier transforms are complex, the duality property often provides Fourier transforms for complex time signals.

10.6 Inverse Fourier Transform

Given a Fourier transform, $X(\omega)$, one approach to computing the corresponding time signal is via the inverse transform formula,

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega$$

Another approach is table lookup, making use of the wide collection of tables of Fourier transform pairs that have been established. However, it turns out that in many situations $X(\omega)$ can be written in the form of a proper rational function in the argument $(j\omega)$, where the term *proper* refers to a rational function in which the degree of the numerator polynomial is no greater than the degree of the denominator polynomial. That is,

$$X(\omega) = \frac{b_n (j\omega)^n + b_{n-1}(j\omega)^{n-1} + \cdots + b_0}{(j\omega)^n + a_{n-1}(j\omega)^{n-1} + \cdots + a_0}$$

This general form is not in the tables, but we can use partial-fraction expansion to write $X(\omega)$ as a sum of the simpler rational functions that are listed in all tables. The linearity property and table lookup then yield the corresponding $x(t)$. To explain some of the mechanics, and mild subtleties that arise, it is convenient to consider simple examples. The first example addresses the issue that often a Fourier transform does not present itself in the nicest form.

Example 1 Given

$$X(\omega) = \frac{1}{-2a\,\omega^2 + j(a^2\omega - \omega^3)}$$

we can use the substitution $\omega^k = (j\omega)^k / j^k$ as follows:

$$X(\omega) = \frac{1}{-2a\frac{(j\omega)^2}{j^2} + j\left(a^2\frac{(j\omega)}{j} - \frac{(j\omega)^3}{j^3}\right)} = \frac{1}{(j\omega)^3 + 2a(j\omega)^2 + a^2(j\omega)}$$

To perform a partial fraction expansion, the denominator polynomial must be put in factored form. Any of the partial-fraction expansion methods can be used, and for many it is convenient to switch from the argument $(j\omega)$ to a more convenient notation. Also, in the table lookup phase, some creativity might be required to recognize the inverse transforms of various terms.

Example 2 To write the Fourier transform in Example 1 in more convenient notation, we substitute s for $(j\omega)$, and proceed as follows (assuming $a \neq 0$):

$$\frac{1}{s^3 + 2a\,s^2 + a^2 s} = \frac{1}{s(s+a)^2} = \frac{1/a^2}{s} - \frac{1/a^2}{s+a} - \frac{1/a}{(s+a)^2}$$

Thus

$$X(\omega) = \frac{1/a^2}{j\omega} - \frac{1/a^2}{a + j\omega} - \frac{1/a}{(a + j\omega)^2}$$

The second and third terms can be found in even the shortest table, if $a > 0$, but the first term might require interpretation. From the standard transforms

$$F[u(t)] = \frac{1}{j\omega} + \pi\,\delta(\omega)\,, \quad F[1] = 2\pi\,\delta(\omega)$$

and the linearity property, we see that $1/(j\omega)$ corresponds to the time signal

$$u(t) - \tfrac{1}{2} = \begin{cases} 1/2\,, & t > 0 \\ -1/2\,, & t < 0 \end{cases} = \tfrac{1}{2}\,\mathrm{sgn}(t)$$

where we have written the result in terms of the so-called *signum* function. Thus we have,

$$x(t) = \tfrac{1}{2a^2}\,\mathrm{sgn}(t) - \tfrac{1}{a^2}e^{-at}u(t) - \tfrac{1}{a}\,t\,e^{-at}u(t)$$

again under the assumption that a is positive.

Perusal of tables of Fourier transforms indicates that most of the simple rational transforms that are covered are strictly-proper rational functions of $(j\omega)$, that is, the numerator degree is strictly less than the denominator degree. This brings up an additional manipulation.

Example 3 Given

$$X(\omega) = \frac{(j\omega)^2 + 2(j\omega) + 2}{(j\omega)^2 + 2(j\omega) + 1}$$

it is convenient to first divide the denominator polynomial into the numerator polynomial to write $X(\omega)$ as a constant plus a strictly-proper rational function. This is another calculation where a change of variable to, say, s instead of $(j\omega)$ might be convenient. In any case, it is easy to verify that the result is

$$X(\omega) = 1 + \frac{1}{(j\omega)^2 + 2(j\omega) + 1} = 1 + \frac{1}{(1 + j\omega)^2}$$

This gives, from standard tables,

$$x(t) = \delta(t) + t\,e^{-t}u(t)$$

Remark We will have a standard table for use in 520.214. This table, reprinted below, and linked on the course webpage, will be provided in exams. Use of any other table is not permitted. That is, anything not on the official table must be derived from the official table or from first principles.

Official 520.214 CT Fourier Transform Table

$x(t)$	$X(\omega)$
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$e^{j\omega_o t}$	$2\pi\delta(\omega - \omega_o)$
$\cos(\omega_o t)$	$\pi\delta(\omega - \omega_o) + \pi\delta(\omega + \omega_o)$
$\sin(\omega_o t)$	$-j\pi\delta(\omega - \omega_o) + j\pi\delta(\omega + \omega_o)$
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$
$e^{-at}u(t),\ a > 0$	$\frac{1}{a + j\omega}$
$te^{-at}u(t),\ a > 0$	$\frac{1}{(a + j\omega)^2}$
$e^{-a\lvert t\rvert},\ a > 0$	$\frac{2a}{a^2 + \omega^2}$
$u(t + T_1) - u(t - T_1)$	$2T_1 \frac{\sin(\omega T_1)}{\omega T_1}$
$\sum_{k=-\infty}^{\infty} \delta(t - kT)$	$\omega_o \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_o)\ ,\ \omega_o = 2\pi / T$

10.7 Fourier Transform and LTI Systems Described by Differential Equations

If a system is described by a first-order, linear differential equation,

$$\dot{y}(t) + ay(t) = bx(t)\,, \quad -\infty < t < \infty$$

then from Section 6.6 we have that the system is linear and time invariant, and the unit-impulse response is given by

$$h(t) = be^{-at}u(t)$$

Therefore we can readily compute the frequency response function of the system,

$$H(\omega) = \frac{b}{a + j\omega}$$

However, this is valid only if the system is stable, that is, $a > 0$. (If $a < 0$, then the Fourier transform of $h(t)$ does not exist, and if $a = 0$, then the Fourier transform has a different form.) Because $H(\omega)$ is a strictly-proper rational function, if the input signal has a proper rational Fourier transform, computation of the response $y(t)$ is simply a matter of computing the inverse Fourier transform of $Y(\omega) = H(\omega)\,X(\omega)$ by partial-fraction expansion and table lookup. That is, for a large class of input signals, the response computation is completely algebraic.

More directly, we can express the relation between time signals in the differential equation as a relation between Fourier transforms. Letting

$$X(\omega) = F[x(t)]\,, \quad Y(\omega) = F[y(t)]$$

linearity and the differentiation property give

$$j\omega Y(\omega) + a\,Y(\omega) = b\,X(\omega)$$

This can be solved algebraically to obtain

$$Y(\omega) = \frac{b}{a + j\omega}\,X(\omega)$$

From this we recognize $H(\omega)$, and the obvious inverse Fourier transform gives $h(t)$, the unit-impulse response of the system. Again, this is valid only for $a > 0$, and a danger is that this condition is not apparent until the inverse Fourier transform is attempted. In other words, the stability condition is not explicit in the algebraic manipulations leading to the frequency response function.

It should be clear that this approach applies to higher-order, linear differential equations that correspond to *stable* systems. The frequency response function in such a case can be written in the form

$$H(\omega) = \frac{b_m(j\omega)^m + \cdots + b_1(j\omega) + b_0}{(j\omega)^n + a_{n-1}(j\omega)^{n-1} + \cdots + a_0}$$

where $m < n$. So, again, computation of the response to a large class of input signals is completely algebraic (though checking the stability condition is more subtle, and is omitted).

Example When the input signal has a Fourier transform that is not a proper rational function, the calculations become slightly more complicated and involve some recognition of combinations of terms. Suppose $a = 2, b = 1$, and the input signal is a unit step function. Then

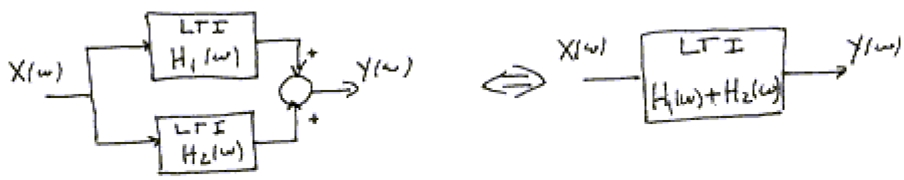
$$Y(\omega) = \frac{1}{2+j\omega}\left[\frac{1}{j\omega} + \pi\,\delta(\omega)\right] = \frac{1}{(j\omega)(2+j\omega)} + \frac{\pi}{2+j\omega}\,\delta(\omega)$$
$$= \frac{1/2}{j\omega} - \frac{1/2}{2+j\omega} + \frac{\pi}{2}\delta(\omega)$$

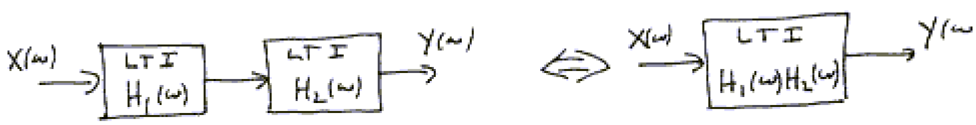
Grouping together the first and last terms, table lookup gives the output signal as

$$y(t) = \tfrac{1}{2}u(t) - \tfrac{1}{2}e^{-2t}u(t)$$

10.8 Fourier Transform and Interconnections of LTI Systems

Interconnections of *stable* LTI systems are conveniently described in terms of frequency response functions, though it must be guaranteed that the overall system also is stable for the overall frequency response function to be meaningful. Assuming this, block diagram equivalences in terms of frequency response functions follow from the time domain results, at least for the first two cases. Namely, for additive parallel connections, where the overall unit-impulse response is the sum of the subsystem unit-impulse responses, and for cascade connections, where the overall unit-impulse response is the convolution of the subsystem unit-impulse responses, we immediately have

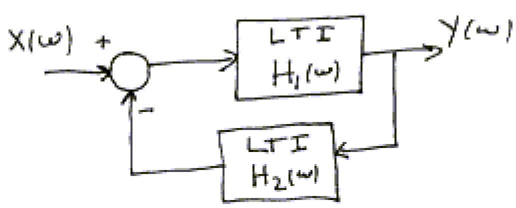




Of course, in these cases it is clear that stability of the overall system follows from stability of the individual subsystems.

The situation is more complicated for the feedback connection of stable LTI systems, but at least the Fourier transform representation permits us to achieve an explicit representation for the overall system, something that we were unable to accomplish in the time domain.

Beginning with the output, the feedback connection below gives the following algebraic relationship between the Fourier transforms of the input and output signals. (Again, the negative sign on the feedback line at the summer is traditional.)

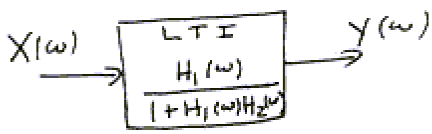


$$
\begin{aligned}
Y(\omega) &= H_1(\omega)\left[X(\omega) - H_2(\omega)Y(\omega)\right] \\
&= H_1(\omega)\,X(\omega) - H_1(\omega)H_2(\omega)Y(\omega)
\end{aligned}
$$

Solving for $Y(\omega)$ by algebraic manipulation gives

$$
Y(\omega) = \frac{H_1(\omega)}{1 + H_1(\omega)H_2(\omega)}\,X(\omega)
$$

That is, the feedback connection above is equivalent to



Of course the overall system, called the *closed-loop system* in this context, must be stable for the frequency response function shown to be meaningful. Unfortunately, the feedback connection of stable systems does not always yield a stable closed-loop system, so that further pursuit of this topic first requires the development of stability criteria for feedback systems.

Example If

$$
H_1(\omega) = \frac{3}{2 + j\omega}\,, \quad H_2(\omega) = k
$$

where k is a constant, then the frequency response of the closed-loop system is

$$
H_{cl}(\omega) = \frac{3/(2 + j\omega)}{1 + 3k/(2 + j\omega)} = \frac{3}{(3k + 2) + j\omega}
$$

This is a valid frequency response if $k > -2/3$, in which case

$$
h_{cl}(t) = 3e^{-(3k+2)t}u(t)
$$

Indeed, by choice of k we can achieve arbitrarily fast exponential decay of the closed-loop system's unit-impulse response! But it is important to note that for $k < -2/3$ the closed-loop system is not stable. Thus it does not have a meaningful frequency response function and the $H_{cl}(\omega)$ we computed is a fiction.

Exercises

1. From the basic definition, compute the Fourier transforms of the signals

(a) $x(t) = e^{-(t-2)}u(t-3)$

(b) $x(t) = e^{-|t+1|}$

(c) $x(t) = \begin{cases} 0, & t \le 0 \\ 2, & 0 < t < 1 \\ 2e^{-(t-1)}, & t \ge 1 \end{cases}$

(d) $x(t) = \sum_{k=0}^{\infty} a^k\,\delta(t-k)\,, \quad |a| < 1$

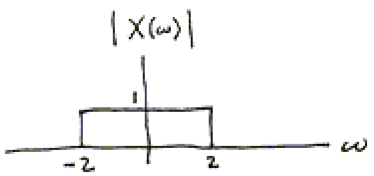
2. From the basic definition, compute the signals corresponding to the Fourier transforms

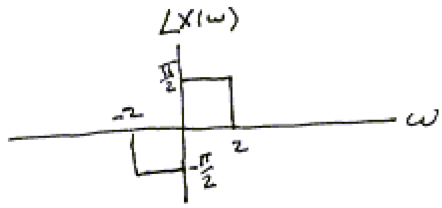
(a) $X(\omega) = 2\pi e^{-|\omega|}$

(b) $|X(\omega)| = \begin{cases} 2\pi, & -2 \le \omega \le 2 \\ 0, & else \end{cases}$ $\qquad \angle X(\omega) = \begin{cases} -\omega, & -2 \le \omega \le 2 \\ 0, & else \end{cases}$

(c) $X(\omega) = e^{-\omega} u(\omega)$

(d) $X(\omega)$ specified by the sketches below:



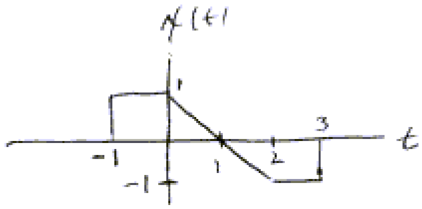


3. Compute the Fourier transforms of the signals

(a) $x(t) = \sum_{k=-\infty}^{\infty} 2(-1)^k \delta(t - 3k)$

(b) $x(t) = \sum_{k=-\infty}^{\infty} e^{-(t-3k)}[u(t-3k) - u(t-3k-1)]$

4. By inspection of the defining formulas for the Fourier transform and inverse Fourier transform, that is, without computing the Fourier transform, evaluate the following quantities for the signal shown below.

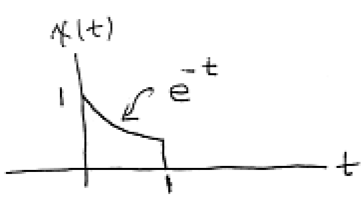


(a) $\int_{-\infty}^{\infty} X(\omega)\, d\omega$

(b) $\int_{-\infty}^{\infty} X(\omega) e^{j\omega}\, d\omega$

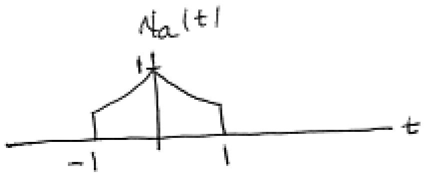
(c) $X(0)$

5. Compute the Fourier transform of the signal $x(t)$ shown below,

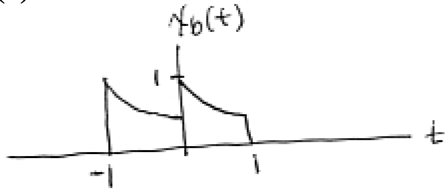


and use the properties of the Fourier transform to determine the transforms of the following signals without calculation.

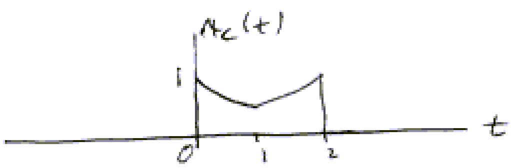
(a)



(b)



(c)

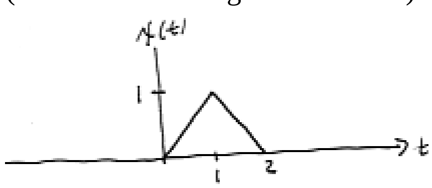


6. A signal $x(t)$ has the Fourier transform

$$X(\omega) = \frac{3 - j\omega}{3 + j\omega}$$

(a) Sketch the magnitude spectrum of the signal.
(b) Sketch the phase spectrum of the signal
(c) Find the signal $x(t)$ by using the properties of the Fourier transform.

7. From the basic definitions and properties of the Fourier transform and inverse Fourier transform, answer the following questions about the transform of the signal shown below (without calculating the transform).



(a) What is $\angle X(\omega)$? (Hint: An even signal has a real Fourier transform.)
(b) What is $X(0)$?
(c) What is $\int_{-\infty}^{\infty} X(\omega)\, d\omega$?

8. Given that the Fourier transform of the signal $x(t) = t\, e^{-2t} u(t)$ is

$$X(\omega) = \frac{1}{(2 + j\omega)^2}$$

sketch the magnitude and phase spectra for the signals

(a) $y(t) = \frac{d}{dt}x(t)$

(b) $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$

(c) $y(t) = x(-2t + 4)$

(d) $y(t) = 2x(t) + \dot{x}(t)$

9. Two LTI systems are specified by the unit-impulse responses $h_1(t) = -2\delta(t) + 5e^{-2t}u(t)$ and $h_2(t) = 2te^{-t}u(t)$. Compute the responses of the two systems to the input signal $x(t) = \cos(t)$.

10. An input signal $x(t)$ applied to the LTI system with frequency response function

$$H(\omega) = \frac{j\omega}{1 + j\omega}$$

yields the output signal

$$y(t) = \delta(t) + 3e^{-t}u(t) - 7e^{-2t}u(t)$$

What is $x(t)$?

11. Suppose $y(t) = x(t)\cos(t)$ and the Fourier transform of $y(t)$ is described in terms of unit-step functions as

$$Y(\omega) = u(\omega + 2) - u(\omega - 2)$$

What is $x(t)$?

12. Suppose $x(t)$ has Fourier transform described in terms of unit-ramp functions as

$$X(\omega) = r(\omega + 1) - 2r(\omega) + r(\omega - 1)$$

and suppose $p(t)$ is periodic with fundamental frequency ω_o and Fourier series coefficients X_k, $k = 0, \pm 1, \pm 2, \ldots$.

(a) If $y(t) = x(t)\,p(t)$, determine an expression for $Y(\omega)$.

(b) Sketch the amplitude spectrum of $y(t)$ if $p(t) = \cos(t/2)$.

(c) Sketch the amplitude spectrum of $y(t)$ if $p(t) = \cos(t)$.

13. Given that the Fourier transform of $x(t) = e^{-|t|}$ is

$$X(\omega) = \frac{2}{1+\omega^2}$$

compute and sketch the magnitude and phase spectra for

$$y(t) = e^{j3t}\,\frac{d}{dt}e^{-|t|}$$

14. Compute the Fourier transform for the signal

$$x(t) = \sin(\omega_o t)\, u(t)$$

(*Hint: You may have to use special properties of impulses in the calculation.*)

15. A continuous-time LTI system is described by the frequency response function

$$H(\omega) = \frac{2}{2 - \omega^2 + j\omega 3}$$

and the input signal has the Fourier transform

$$X(\omega) = e^{-j\omega 3}$$

Compute the response $y(t)$.

16. Use partial fraction expansion to compute the inverse Fourier transform for

(a) $X(\omega) = \dfrac{5j\omega + 12}{(j\omega)^2 + 5j\omega + 6}$

(b) $X(\omega) = \dfrac{4}{3 - \omega^2 + 4j\omega}$

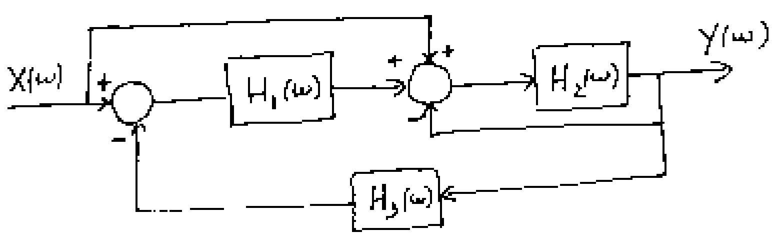
17. Compute the inverse Fourier transform for

(a) $X(\omega) = \dfrac{e^{j(\pi - \pi\omega)}(5 - \omega^2 + j4\omega)}{(9 - \omega^2 + j6\omega)(2 + j\omega)}$

(b) $X(\omega) = e^{-j2\omega} + \dfrac{10 + 10e^{-j\omega}}{2 - \omega^2 + j3\omega}$

18. Compute the overall frequency response function $H(\omega) = Y(\omega) / X(\omega)$ for the systems shown below. (Of course, assume that the subsystems and the overall system are stable.)

(a)



Notes for Signals and Systems

11.1 Introduction to the Unilateral Laplace Transform

From Chapter 10 it is clear that there is one main limitation on the use of the Fourier transform: the signal must be such that the transform integral converges, or such that we can apply generalized function techniques to arrive at a transform (for example, the case of periodic signals). It turns out that this limitation can be avoided, particularly for right-sided signals, by including a damping factor in the integral. The resulting transform is called the *unilateral Laplace transform*.

For a right-sided, or *unilateral*, signal $x(t)$, sometimes written as $x(t)u(t)$ to emphasize that the signal is zero for $t < 0$, the unilateral Laplace transform is defined as

$$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$$

Here s is a complex variable, often written in rectangular form using the standard notation $s = \sigma + j\omega$. The lower limit of integration is shown as 0^- to emphasize the fact that an impulse or doublet at $t = 0$ is included in the range of integration, but often we leave this understood and simply write the lower limit as 0.

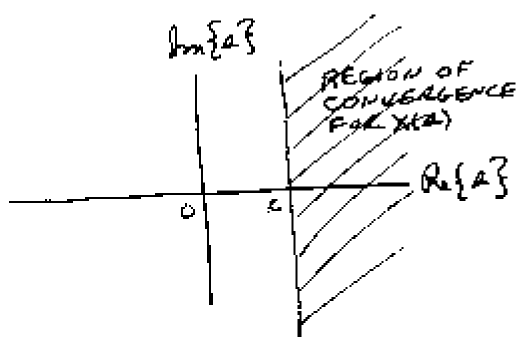
If we consider $\sigma = \text{Re}\{s\} > 0$, then

$$|e^{-st}| = |e^{-\sigma t}e^{-j\omega t}| = |e^{-\sigma t}| \to 0 \;\; as\; t \to \infty$$

Therefore $X(s)$ can be well defined even though $x(t)$ does not go to zero as t goes to ∞. Indeed, if $x(t)$ is of "exponential order," that is, there exist real constants K, c such that

$$|x(t)| \le Ke^{ct}\;, \quad t \ge 0$$

then $X(s)$ exists if we think of s as satisfying $\text{Re}\{s\} > c$. In other words, for signals of exponential order the unilateral Laplace transform always exists for a half-plane of complex values of s, as shown below.



Because signals encountered in the sequel will always be of exponential order, we can be a bit cavalier and ignore detailed analysis of the regions of convergence, confident in the knowledge that there is a whole half-plane of values of s for which $X(s)$ is well defined. And the actual numerical values of s for which the integral converges turn out to be of no interest for our purposes.

Example The signal

$$x(t) = e^{t^2} u(t)$$

is not of exponential order since for any given values of K and c,

$$e^{t^2} > Ke^{ct} \ , \quad \text{for } t \text{ sufficiently large}$$

The signal,

$$x(t) = e^{5t} u(t)$$

is of exponential order, and we can take $K = 1$, $c = 5$ to prove it.

Example The Laplace transform of $x(t) = e^{-3t}u(t)$ is

$$X(s) = \int_{0^-}^{\infty} e^{-3t} u(t) e^{-st} dt = \int_{0}^{\infty} e^{-(s+3)t} dt$$

$$= \frac{-1}{s+3} e^{-(s+3)t} \Big|_0^{\infty} = \frac{1}{s+3}$$

Here a half-plane of convergence is given by $\mathrm{Re}\{s\} > -3$, and indeed this condition is crucial in evaluating the integral at the upper limit. For the signal $x(t) = e^{3t}u(t)$, a similar calculation gives

$$X(s) = \frac{1}{s-3}$$

and the half-plane of convergence in this case is $\mathrm{Re}\{s\} > 3$. However, as mentioned above, we will not insist on keeping track of the convergence region.

For signals involving generalized functions, the notion of exponential order does not apply, but in typical cases the special properties of generalized functions can be used to evaluate the Laplace transform.

Example The Laplace transform of the impulse, $x(t) = \delta(t)$, is easily evaluated using the sifting property:

$$X(s) = \int_{0^-}^{\infty} \delta(t) e^{-st} dt = e^{-s0} = 1$$

Example The Laplace transform of the unit step function is

$$X(s) = \int_{0^-}^{\infty} u(t) e^{-st} dt = \int_{0}^{\infty} e^{-st} dt = \frac{1}{s}$$

where in this case a half-plane of convergence is given by $\mathrm{Re}\{s\} > 0$.

Remark If a right-sided signal $x(t)$ has a unilateral Laplace transform that converges for $\mathrm{Re}\{s\} = 0$, then we can write, taking $\sigma = 0$,

$$X(s) = \int_{0}^{\infty} x(t) e^{-st} dt = \int_{0}^{\infty} x(t) e^{-j\omega t} dt = F[x(t)]$$

That is, the Laplace transform with $\sigma = 0$ is the Fourier transform for right-sided signals. (Sometimes this is written as

$$X(s)\big|_{s=j\omega} = X(j\omega)$$

which leads to a different notation for the Fourier transform than we have used. Namely, we write the Fourier transform as $X(\omega)$, rather than $X(j\omega)$, absorbing the imaginary unit j into the function rather than displaying it in the argument. This unfortunate notational collision should be viewed as a mild inconvenience, and it should not be permitted to obscure the relationship between the Fourier and Laplace transforms of right-sided signals.)

Example For $x(t) = e^{-3t}u(t)$ we see that the half-plane of convergence includes $\sigma = 0$, and from above we have

$$X(\omega) = \frac{1}{3 + j\omega}$$

For the unit-step function we have $X(s) = 1/s$, but in this case the region of convergence does *not* include $\sigma = 0$, and indeed the Fourier transform of the unit step is not simply $1/(j\omega)$.

11.2 Properties of the Unilateral Laplace Transform

We now consider a variety of familiar operations on a right-sided signal $x(t)$, and interpret the effect of these operations on the corresponding unilateral Laplace transform $X(s)$. Of course, the operations we consider must yield right-sided signals. We assume that signals are of exponential order so that existence of the Laplace transform is assured. Furthermore, we should verify that each operation considered yields a signal that also is of exponential order. Often this is obvious, and will not be mentioned, but care is needed in a couple of cases.

Throughout we use the following notation for the Laplace transform where *L* denotes a "Laplace transform operator:"

$$X(s) = L[x(t)] = \int_0^\infty x(t)e^{-st}\,dt$$

Linearity If $L[x(t)] = X(s)$ and $L[z(t)] = Z(s)$, then for any constant *a*,

$$L[a\,x(t) + z(t)] = a\,X(s) + Z(s)$$

This property follows directly from the definition.

Time Delay If $L[x(t)] = X(s)$, then for any constant $t_o \ge 0$,

$$L[x(t - t_o)u(t - t_o)] = e^{-st_o}X(s)$$

The calculation verifying this is by now quite standard. Begin with

$$L[x(t - t_o)u(t - t_o)] = \int_0^\infty x(t - t_o)u(t - t_o)e^{-st}\,dt$$

and change integration variable from t to $\tau = t - t_o$ to obtain the result. Notice that we use the unit-step notation to make explicit the fact that the right-shifted signal, $x(t - t_o)$ is zero for $t < t_o$.

Example For a rectangular pulse, $x(t) = Ku(t) - Ku(t - t_o)$, $t_o > 0$, we can use linearity and the delay property to write

$$X(s) = \frac{K}{s} - e^{-t_o s}\frac{K}{s} = K\,\frac{1 - e^{-t_o s}}{s}$$

Time Scaling If $L[x(t)] = X(s)$, then for any constant $a > 0$,

$$L[x(at)] = \tfrac{1}{a} X\left(\tfrac{s}{a}\right)$$

This is another familiar calculation, and the details will be skipped. The assumption that $a > 0$ is required so that the scaled signal is right sided.

The next two properties require a more careful interpretation of the lower limit in the Laplace transform definition, and we write that limit as 0^-.

Differentiation If $L[x(t)] = X(s)$, and the time-derivative signal $\dot{x}(t)$ has a Fourier transform, then

$$L[\dot{x}(t)] = sX(s) - x(0^-)$$

To justify this property, directly compute the transform, using integration-by-parts:

$$L[\dot{x}(t)] = \int_{0^-}^{\infty} \dot{x}(t)e^{-st}dt = x(t)e^{-st}\Big|_{0^-}^{\infty} + \int_{0^-}^{\infty} x(t)\, s\, e^{-st}dt$$

We assume that $\sigma = \mathrm{Re}\{s\}$ is such that $x(t)e^{-st} \to 0$ as $t \to \infty$, and we interpret $x(0^-)e^{-s0^-}$ as $x(0^-)$ to arrive at the claimed result.

Example Beginning with $L[u(t)] = 1/s$, the differentiation property confirms that

$$L[\delta(t)] = L[\dot{u}(t)] = s\frac{1}{s} = 1$$

Note that we can iterate the differentiation property to obtain the Laplace transform for higher derivatives, for example,

$$L[\ddot{x}(t)] = s\,L[\dot{x}(t)] - \dot{x}(0^-) = s^2 L[x(t)] - s\,x(0^-) - \dot{x}(0^-)$$

Integration If $L[x(t)] = X(s)$, and

$$z(t) = \int_{0^-}^{t} x(\tau)d\tau\, u(t)$$

where the unit-step is appended simply for emphasis, then

$$Z(s) = \frac{1}{s} X(s)$$

The proof of this is another integration by parts that is outlined below:

$$L[z(t)] = \int_{0^-}^{\infty} \int_{0^-}^{t} x(\tau)\, d\tau\, u(t)\, e^{-st} dt = \int_{0^-}^{t} x(\tau)\, d\tau\, u(t) \frac{-1}{s} e^{-st} \Bigg|_{0^-}^{\infty} + \int_{0^-}^{\infty} x(t) \frac{1}{s} e^{-st} dt$$

$$= \frac{1}{s} X(s)$$

The evaluations of the first term resulting from the integration-by-parts are both zero, but for different reasons. It can be shown that the running integral of a signal of exponential order is of exponential order, and so we can assume that $\sigma = \mathrm{Re}\{s\}$ is such that as $t \to \infty$ the product of the running integral and the exponential goes to zero. The evaluation at $t = 0^-$ yields zero for more obvious reasons.

Convolution If $x(t)$ and $h(t)$ are right-sided signals with unilateral Laplace transforms $X(s)$ and $Z(s)$, then the convolution $(h * x)(t)$ yields a right-sided signal that can be written

$$y(t) = \int_{0}^{t} x(\tau) h(t-\tau)\, d\tau\, u(t)$$

with Laplace transform

$$Y(s) = H(s) X(s)$$

The proof of this property is very similar to the Fourier-transform case, and therefore is omitted.

Final Value Theorem If $L[x(t)] = X(s)$ and the limits

$$\lim_{t\to\infty} x(t)\,, \quad \lim_{s\to 0} sX(s)$$

both exist, then

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

Rather than prove this result, we present an example that illustrates the danger in applying it recklessly.

Example A straightforward calculation gives

$$L[\sin(t)u(t)] = \int_{0}^{\infty} \sin(t) e^{-st} dt = \int_{0}^{\infty} \frac{e^{jt} - e^{-jt}}{2j} e^{-st} dt$$

$$= \frac{-1/2j}{s-j} e^{-(s-j)t} \Bigg|_{0}^{\infty} + \frac{1/2j}{s+j} e^{-(s+j)t} \Bigg|_{0}^{\infty}$$

$$= \frac{1}{s^2+1}$$

Obviously,

$$\lim_{s\to 0} s\, X(s) = \lim_{s\to 0} \frac{s}{s^2+1} = 0$$

However, $\lim_{t\to\infty}[\sin(t)u(t)]$ does not exist and we see that the Final Value Theorem can give misleading results when loosely applied!

11.3 Inverse Unilateral Laplace Transform

Inspection of the Laplace transforms we have computed, or a table of transforms, indicates that the signals typically encountered have transforms that are strictly-proper rational functions. These are ratios of polynomials in s with the degree of the numerator polynomial less than the degree of the denominator polynomial. As might be expected from the Fourier-transform case, partial fraction expansion, followed by table lookup, is the main tool for computing the time signal corresponding to a given transform. (There is a more general inverse transform formula, but it involves line integrals in the complex plane and we will not make use of it.)

Remark Shown below is the standard table of Laplace transforms for use in 520.214. Use of any other table in exams or homework assignments is not permitted. Anything not on this official table must be derived from entries on the table or from the definition of the transform.

Official 520.214 Unilateral Laplace Transform Table

$x(t)$	$X(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$r(t)$	$\frac{1}{s^2}$
$e^{-at}u(t)$	$\frac{1}{s+a}$
$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$
$\cos(\omega_o t)u(t)$	$\frac{s}{s^2+\omega_o^2}$
$\sin(\omega_o t)u(t)$	$\frac{\omega_o}{s^2+\omega_o^2}$
$e^{-at}\cos(\omega_o t)u(t)$	$\frac{s+a}{(s+a)^2+\omega_o^2}$
$e^{-at}\sin(\omega_o t)u(t)$	$\frac{\omega_o}{(s+a)^2+\omega_o^2}$

We illustrate the calculation of inverse transforms with two examples.

Example Given

$$X(s) = \frac{s^2+s+1}{s^2+1}$$

which is a proper, but not strictly-proper, rational function, we can divide the numerator by the denominator to write

$$X(s) = 1 + \frac{s}{s^2 + 1}$$

Using linearity of the Laplace transform, we can treat the terms separately. Partial fraction expansion of the second term gives

$$\frac{s}{s^2+1} = \frac{s}{(s+j)(s-j)} = \frac{1/2}{s+j} + \frac{1/2}{s-j}$$

From the table of transforms,

$$L^{-1}\left[\frac{s}{s^2+1}\right] = \tfrac{1}{2}e^{-jt}u(t) + \tfrac{1}{2}e^{jt}u(t) = \cos(t)\,u(t)$$

Therefore

$$x(t) = \delta(t) + \cos(t)\,u(t)$$

Another case that is straightforward to handle is when there are "delay factors" in the transform.

Example Given

$$X(s) = \frac{s\,e^{-2s} + e^{-s}}{s^2+1}$$

we can write

$$X(s) = e^{-2s}\,\frac{s}{s^2+1} + e^{-s}\,\frac{1}{s^2+1}$$

Using the linearity, delay, and derivative properties in conjunction with the previous example, we obtain

$$x(t) = \cos(t-2)\,u(t) + \sin(t-1)\,u(t-1)$$

11.4 Systems Described by Linear Differential Equations

Consider a system where the input and output signals are related by the first-order differential equation

$$\dot{y}(t) + ay(t) = bx(t)$$

Assuming that the input signal is right sided, and assuming that initial condition at $t = 0$ is zero, the output signal is right sided and the system is linear and time invariant. (In particular, since an LTI system with identically zero input must have identically zero output, the assumption of zero initial condition is important.)

In the setting of right-sided input and output signals, the system can be described in terms of unilateral Laplace transforms. Regardless of the values of the constants a and b, and in particular regardless of the stability property of the system, we can compute the Laplace transform of the unit impulse response

$$h(t) = be^{-at}u(t)$$

to obtain

$$H(s) = \frac{b}{s+a}$$

Rather than the term frequency response function, this is called the *transfer function* of the system, and in terms of the Laplace transforms $X(s)$ and $Y(s)$ of the right-sided input and output signals the system is described by

$$Y(s) = H(s)X(s)$$

Another approach is to equate the Laplace transforms of the left and right sides of the differential equation, and this approach has the advantage of not requiring knowledge of the unit-impulse response. Using the linearity and differentiation properties gives

$$(s+a)Y(s) = bX(s)$$

Thus, again, we obtain

$$\frac{Y(s)}{X(s)} = H(s) = \frac{b}{s+a}$$

If the input signal has a proper rational Laplace transform, then it is clear that the output signal has a strictly-proper rational Laplace transform. Therefore we can solve for the response to a wide class of input signals by the algebraic process of partial fraction expansion and table lookup.

Again, an advantage of the Laplace-transform approach in this unilateral setting is that systems with unbounded input signals and/or output signals, or systems that are unstable, can be treated, in contrast to the Fourier transform approach.

Example For the case where $a = -1$, $b = 1$ and where the input signal is

$$x(t) = e^{3t}u(t)$$

that is, an unstable system with unbounded input signal, we immediately obtain

$$Y(s) = \frac{1}{(s-1)(s-3)}$$

Partial fraction expansion easily leads to

$$y(t) = -\tfrac{1}{2}e^{t}u(t) + \tfrac{1}{2}e^{3t}u(t)$$

For systems described by higher order linear differential equations, again with unilateral input and output signals and zero initial conditions,

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_0 y(t) = b_{n-1}x^{(n-1)}(t) + \cdots + b_0 x(t)$$

it is straightforward to equate the Laplace transforms of the right and left sides to show that the corresponding transfer function is

$$H(s) = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

This is a strictly-proper rational function of s. Thus for input signals that have proper rational Laplace transforms, the output signal will have a proper rational Laplace transform, and the solution procedure for the output signal is again algebraic, though of course the roots of the denominator must be computed for the partial fraction expansion.

11.5 Introduction to Laplace Transform Analysis of LTI Systems

We consider LTI systems with right-sided input signals in this section, and furthermore we assume that the system transfer function is a strictly-proper rational function. Thus we can think of the system as arising from a differential equation description, though that is not necessary. In any case, we introduce some standard methods based on the transfer function description of the system.

When we write such a transfer function, or more generally any strictly-proper Laplace transform,

$$H(s) = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

we will assume that there are no common roots of the numerator and denominator polynomials. That is, the numerator and denominator polynomials are assumed to be relatively prime. This assumption is made to avoid equivalent forms of the transfer function or transform that superficially appear different.

Definition The *poles* of a rational transfer function (or transform) are the roots of the denominator polynomial, and the *zeros* are the roots of the numerator polynomial.

In counting the poles and zeros, we use the standard terminology associated with repeated roots of a polynomial.

Example The transfer function

$$H(s) = \frac{5(s+2)}{s^3 + 4s^2 + 5s + 2} = \frac{5(s+2)}{(s+2)(s+1)^2} = \frac{5}{(s+1)^2}$$

has no zeros and two poles at $s = -1$ (often stated as a multiplicity 2 pole at $s = -1$) .

There are two important results that can be stated immediately using this definition, and the proofs are essentially obvious applications of partial fraction expansion and inspection of the transform table for the types of terms that arise from partial fraction expansion.

Theorem A right-sided signal $x(t)$ with strictly-proper rational Laplace transform is bounded if and only if all poles of the transform have non-positive real parts and those with zero real parts have multiplicity 1.

Theorem An LTI system with strictly-proper rational transfer function is stable if and only if all poles of the transfer function have negative real parts.

Example Consider a system with transfer function

$$H(s) = \frac{20}{s(s+2)}$$

and suppose the input signal is a unit-step function,

$$X(s) = \frac{1}{s}$$

Then

$$Y(s) = \frac{20}{s^2(s+2)} = \frac{10}{s^2} - \frac{5}{s} + \frac{5}{s+2}$$

and the output signal is given by

$$y(t) = 10\,r(t) - 5u(t) + 5e^{-2t}u(t)$$

This response is unbounded, because of the ramp component, which is not unexpected since the system is not stable. Notice, however, that among our typical input signals the only bounded input signal that produces an unbounded response is a step input.

Example Consider a system with transfer function

$$H(s) = \frac{s-3}{(s+1)(s+2)}$$

and suppose the input signal is described by

$$X(s) = \frac{1}{s-3}$$

that is, the input is the unbounded signal $x(t) = e^{3t}u(t)$. Then

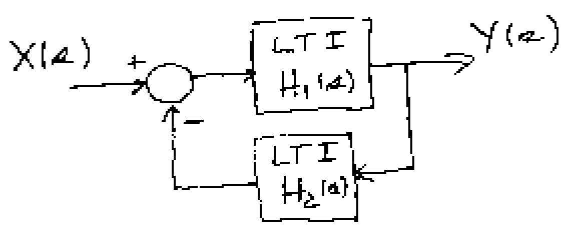
$$Y(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

and the output is the bounded signal

$$y(t) = e^{-t}u(t) - e^{-2t}u(t)$$

This example gives an interpretation of the zeros of a transfer function in terms of growing exponential inputs that are "swallowed" by the system!

Remark It should be clear that the transfer function description of interconnections of LTI systems is very similar in appearance to the frequency response function description based on the Fourier transform discussed in Section 10.8. However, the Laplace transform approach is not beset by the stability limitations that are implicit in applying the Fourier transform. Consider the feedback system shown below, where $H_1(s)$ and $H_2(s)$ are the subsystem transfer functions.



Straightforward calculations give the unsurprising result that the overall system is described by the transfer function

$$H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

Under reasonable hypotheses we can show that $H(s)$ is a strictly-proper rational function as follows. Suppose that $H_1(s)$ is a strictly-proper rational function and $H_2(s)$ is a proper rational function. Then we can write these transfer functions in terms of their numerator and denominator polynomials as

$$H_1(s) = \frac{n_1(s)}{d_1(s)} \quad , \quad H_2(s) = \frac{n_2(s)}{d_2(s)}$$

where $\deg n_1(s) < \deg d_1(s)$ and $\deg n_2(s) \le \deg d_2(s)$. Then, in polynomial form,

$$H(s) = \frac{n_1(s)d_2(s)}{d_1(s)d_2(s) + n_1(s)n_2(s)}$$

and it follows that $H(s)$ is a strictly-proper rational function. Thus regardless of stability issues, the response to the overall system to various input signals can be calculated in the usual way.

Example Repeating the example from Section 10.8, if

$$H_1(s) = \frac{3}{s+2} \quad , \quad H_2(s) = k$$

then

$$H(s) = \frac{3}{s+2+3k}$$

If the input to the feedback system is a unit step, $X(s) = 1/s$, then the response is described by

$$Y(s) = \frac{3}{s(s+2+3k)}$$

If $k = -2/3$, then $Y(s) = 3/s^2$ and $y(t) = 3r(t)$. Otherwise an easy partial fraction expansion calculation gives

$$y(t) = \tfrac{3}{2+3k}\,u(t) - \tfrac{3}{2+3k}\,e^{-(2+3k)t}u(t)$$

Inspection of these responses indicates, and the stability criterion described above confirms, that the system is stable if and only if $2+3k > 0$. However, in any case the transfer function representation is valid and can be used for response calculations and other purposes.

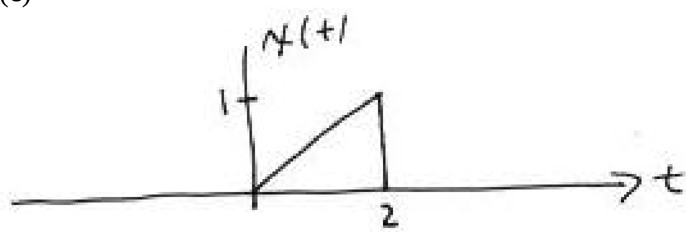
Exercises

1. Using either the basic definition or tables and properties, compute the Laplace transforms of

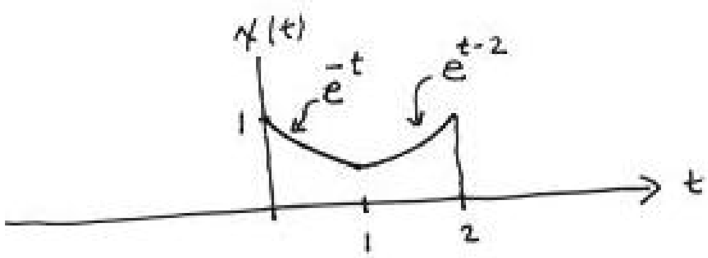
(a) $x(t) = e^{-2t}u(t-1)$

(b) $x(t) = \delta(t-1) + \delta(t) + e^{-2(t+3)}u(t-1)$

(c)



(d)



2. For each of the following Laplace transforms, use the final value theorem to determine $\lim_{t\to\infty} x(t)$, and state whether the conclusion is valid.

(a) $X(s) = \dfrac{2s+4}{s^2+5s+6}$

(b) $X(s) = \dfrac{2s+4}{s^3+5s^2+6s}$

(c) $X(s) = \dfrac{e^{-s}}{s^3+5s^2+6s}$

(d) $X(s) = \dfrac{2}{(s-1)^2}$

3. Given that the Laplace transform of $x(t) = \cos(2t)u(t)$ is $X(s) = \dfrac{s}{s^2+4}$ compute the Laplace transform of $dx(t)/dt$ by two methods: First, differentiate the signal and use the tables. Second, use the differentiation property.

4. Determine the final value of the signal $x(t)$ corresponding to

(a) $X(s) = \dfrac{2s^2+3}{s^2+5s+6}$

(b) $X(s) = \dfrac{2s^2+3}{s^3+5s^2+6s}$

(c) $X(s) = \dfrac{3}{s^2-1}$

5. For the system with transfer function

$$\frac{Y(s)}{X(s)} = \frac{s-3}{(s+4)^2}$$

compute the steady-state response $y_{ss}(t)$, the time function that $y(t)$ approaches asymptotically, as $t \to \infty$, to the input signals

(a) $x(t) = e^{-3t}\,u(t)$ (b) $x(t) = e^{3t}\,u(t)$ (c) $x(t) = 2\sin(3t)u(t)$

(d) $x(t) = \delta(t)$ (e) $x(t) = u(t)$

(Hint: Do not calculate quantities you don't need! And you may skip calculation of the phase angle in (c) – simply call it θ.)

6. Compute the signal $x(t)$ corresponding to

(a) $X(s) = \dfrac{5s+4}{s^3+3s^2+2s}$

(b) $X(s) = \dfrac{e^{-4s}(s+1)}{s^3 - s}$

(c) $X(s) = \dfrac{10(s+2)}{(s+5)(s^3 + 2s^2 + s)}$

7. Find a differential equation description for the LTI system described by

(a) $H(s) = \dfrac{3s}{(s+1)(s+2)}$

(b) $h(t) = [2 - 2e^{-t}]u(t)$

8. An LTI system with input signal

$$x(t) = [e^t + e^{-2t}]u(t)$$

has the response

$$y(t) = [2 - 3t]e^{-2t}u(t)$$

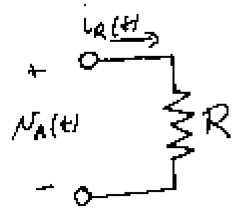
What is the transfer function, $H(s)$, of the system? What is the unit-impulse response of the system?

Notes for Signals and Systems

12.1 Unilateral Laplace Transform – Application to Circuits

When considering RLC circuits, one approach is to write the differential equation (or integro-differential equation) for the circuit, and then solve the equation using the Laplace transform. However, a typically more efficient approach is to consider the circuit directly in terms of Laplace transform representations. We assume in doing this that the input voltage or current for the circuit is a right-sided signal. For the case of zero initial conditions, this permits describing the behavior of each circuit element in terms of a transfer function, as defined in terms of the unilateral Laplace transform..

A resistor, as shown below, is the simplest case.



The voltage-current relation is

$$v_R(t) = R\, i_R(t)\,, \quad t \geq 0$$

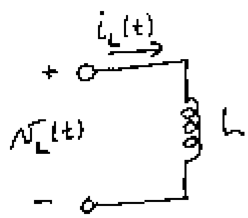
and this can be represented as a "resistance transfer function,"

$$\frac{V_R(s)}{I_R(s)} = R$$

or a "conductance transfer function,"

$$\frac{I_R(s)}{V_R(s)} = \frac{1}{R}$$

For an inductor,



the voltage-current relation is

$$v_L(t) = L\frac{di_L(t)}{dt}\,, \quad t \geq 0$$

With zero initial conditions, this gives the transfer function descriptions

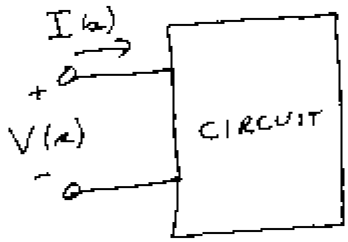
$$\frac{V_L(s)}{I_L(s)} = Ls$$

and

$$\frac{I_L(s)}{V_L(s)} = \frac{1}{Ls}$$

The terminology that goes with these transfer functions can be formally defined as follows.

Definition For a two-terminal electrical circuit



with all independent sources in the circuit set to zero and all initial conditions zero, the *impedance* $Z(s)$ of the circuit is the transfer function

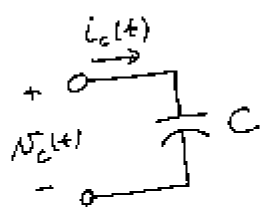
$$Z(s) = \frac{V(s)}{I(s)}$$

and the *admittance* $Y(s)$ of the circuit is the transfer function

$$Y(s) = \frac{I(s)}{V(s)}$$

Using this definition, the impedance of a resistor is the resistance, and the impedance of an inductor is $Z(s) = Ls$. Or, the admittance of a resistor is the conductance, $1/R$, and the admittance of an inductor is $Y(s) = 1/(Ls)$.

For a capacitor,



with voltage-current relation

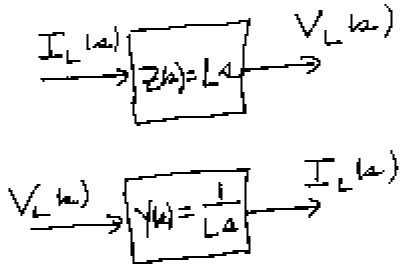
$$v_C(t) = \frac{1}{C}\int_0^t i_C(\tau)\, d\tau$$

we have

$$V_C(s) = \frac{1}{Cs}\, I_C(s)$$

and thus the impedance is $Z(s) = 1/(Cs)$ while the admittance is $Y(s) = Cs$.

For each of the three basic circuit elements, we can represent the voltage current behavior in the Laplace-transform domain by a block diagram, with appropriate labels depending on the choice of input or output. For example,



Once we represent a circuit in this way, a key observation is that the basic circuit analysis tools such as Kirchhoff's laws can be applied in the transform domain. If $i_1(t), \ldots, i_K(t)$ are the currents entering a node, then the current law

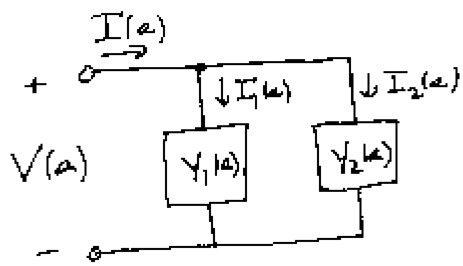
$$\sum_{k=1}^{K} i_k(t) = 0$$

can alternately be expressed in the transform domain as

$$\sum_{k=1}^{K} I_k(s) = 0$$

Obviously a similar statement can be made about the sum of voltages across a number of circuit elements in a loop. This means that circuit analysis in the Laplace domain proceeds much as does resistive circuit analysis in the time domain, namely, it is algebraic in nature with impedance and admittance of circuit elements playing roles similar to resistance and conductance.

Example Consider the parallel connection of circuit elements described by their admittances, as shown below.



Kirchhoff's current law at the top node gives

$$\begin{aligned} I(s) &= I_1(s) + I_2(s) \\ &= Y_1(s)V(s) + Y_2(s)V(s) \\ &= [Y_1(s) + Y_2(s)]V(s) \end{aligned}$$

Thus the admittance of the overall circuit is

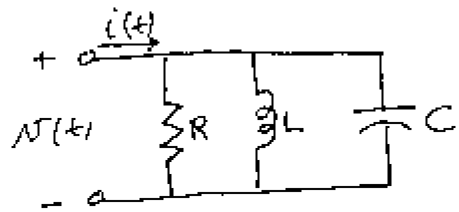
$$Y(s) = \frac{I(s)}{V(s)} = Y_1(s) + Y_2(s)$$

It is easy to see that this calculation extends to any number of admittances in parallel, and leads to the statement that "admittances in parallel add." Form the circuit admittance, it is an easy calculation to obtain the impedance:

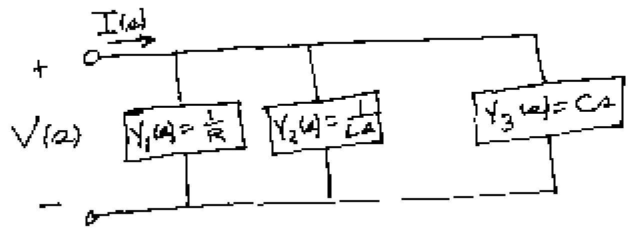
$$Z(s) = \frac{V(s)}{I(s)} = \frac{1}{Y_1(s) + Y_2(s)} = \frac{1}{\frac{1}{Z_1(s)} + \frac{1}{Z_2(s)}} = \frac{Z_1(s)Z_2(s)}{Z_1(s) + Z_2(s)}$$

These expressions have familiar forms from the analysis of resistive circuits when we interpret impedance as resistance and admittance as conductance.

Example To calculate the impedance of the circuit



we first redraw it as a Laplace domain diagram with admittance labels:



Then

$$Y(s) = \frac{I(s)}{V(S)} = Y_1(s) + Y_2(s) + Y_3(s) = \frac{1}{R} + \frac{1}{Ls} + Cs$$

and the circuit impedance is

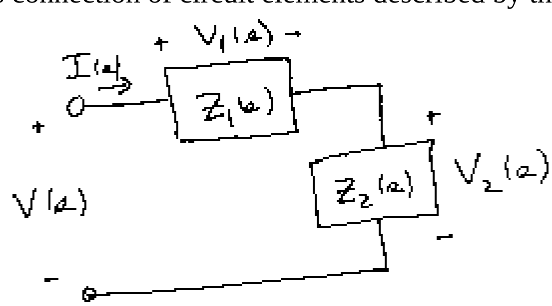
$$Z(s) = \frac{1}{\frac{1}{R} + \frac{1}{Ls} + Cs} = \frac{\frac{1}{C}s}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

Notice that if $R, L, C > 0$, the usual case, then the circuit impedance is a (bounded-input, bounded-output) stable system since the poles of $Z(s)$ will have negative real parts. Of course, to compute the voltage for a given current $i(t)$, we compute $I(s)$ and then the terminal voltage (Laplace transform) is given by

$$V(s) = Z(s)I(s)$$

Finally, $v(t)$ can be computed by partial fraction expansion and table lookup, assuming that $I(s)$ is a proper rational function.

Example Consider a series connection of circuit elements described by their impedances,



From Kirchhoff's voltage law,

$$V(s) = V_1(s) + V_2(s) = Z_1(s)I(s) + Z_2(s)I(s)$$
$$= [Z_1(s) + Z_1(s)]I(s)$$

Therefore the impedance of the circuit is

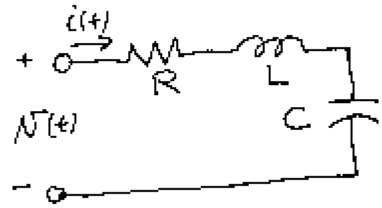
$$Z(s) = \frac{V(s)}{I(s)} = Z_1(s) + Z_2(s)$$

and the admittance is

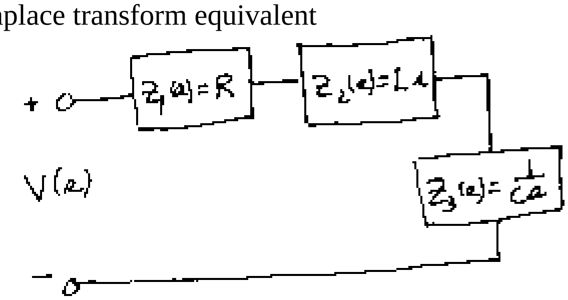
$$Y(s) = \frac{I(s)}{V(s)} = \frac{1}{Z_1(s) + Z_2(s)}$$

This calculation extends to a series connection of any number of circuit elements in the obvious manner.

Example The impedance of the circuit



is computed from the Laplace transform equivalent

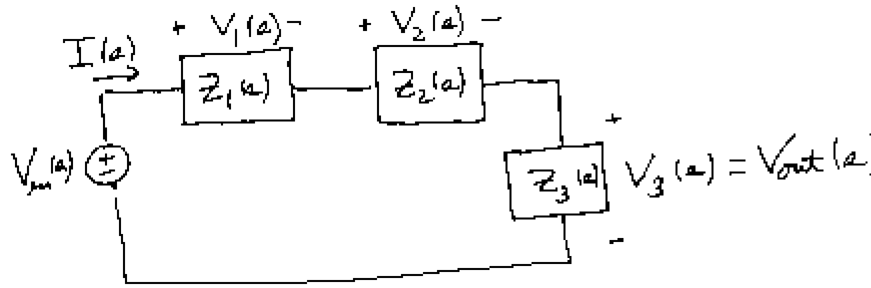


as

$$Z(s) = R + Ls + \frac{1}{Cs} = \frac{Ls^2 + Rs + 1/C}{s}$$

It is interesting to note that this impedance is an “improper” rational function of s, in that the numerator degree is higher than the denominator degree. For example, a unit-step in current to the circuit will produce an impulse in voltage. Also, the circuit is unstable since $Z(s)$ has a pole at $s = 0$. In particular, a unit step in current produces a ramp in voltage.

The analysis of circuits via the transformed circuit also applies to other transfer functions of interest, in addition to the impedance and admittance. That is, the techniques apply for other choices of input and output signals, with, of course, all initial conditions zero. Consider the circuit



where the voltage transfer function $V_{out}(s)/V_{in}(s)$ is of interest. Clearly

$$V_{in}(s) = V_1(s) + V_2(s) + V_3(s) = [Z_1(s) + Z_2(s) + Z_3(s)]I(s)$$

which gives

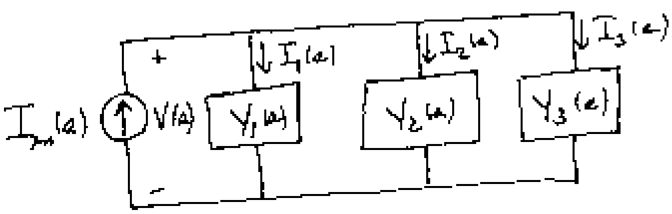
$$I(s) = \frac{1}{Z_1(s) + Z_2(s) + Z_3(s)} V_{in}(s)$$

Then the output voltage transform is

$$V_{out}(s) = Z_3(s)I(s) = \frac{Z_3(s)}{Z_1(s) + Z_2(s) + Z_3(s)} V_{in}(s)$$

and the voltage transfer function is clear. Clearly this is the analog of the resistive *voltage divider circuit.*

In a similar manner, the current divider circuit



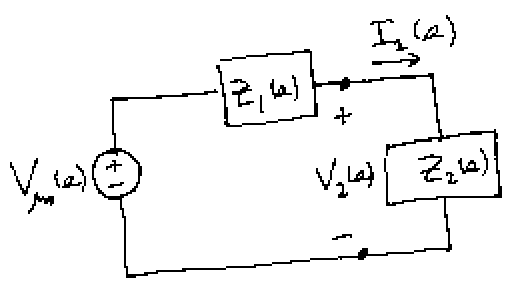
yields transfer functions from the input current to the j^{th} − branch current via the calculations

$$I_{in}(s) = I_1(s) + I_2(s) + I_3(s) = [Y_1(s) + Y_2(s) + Y_3(s)]V_{in}(s)$$

yielding

$$I_j(s) = Y_j(s)V_{in}(s) = \frac{Y_j(s)}{Y_1(s) + Y_2(s) + Y_3(s)} I_{in}(s)$$

The notion of source transformations in resistive circuits also can be applied in the setting of the transformed circuit. These transformations replace voltage sources in series with impedances by current sources in parallel with impedances, or the reverse. For the circuit shown below,



the voltage divider rule immediately gives

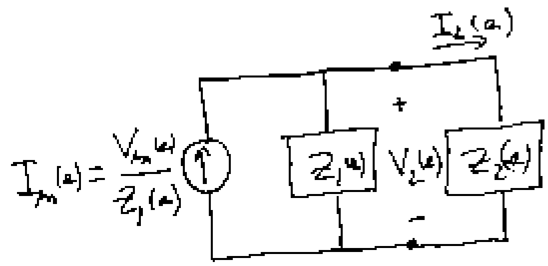
$$V_2(s) = \frac{Z_2(s)}{Z_1(s) + Z_2(s)} V_{in}(s)$$

Assuming $Z_1(s) \neq 0$, this expression can be rearranged as

$$V_2(s) = \frac{Z_1(s)Z_2(s)}{Z_1(s) + Z_2(s)} \frac{V_{in}(s)}{Z_1(s)} = \frac{1}{\frac{1}{Z_1(s)} + \frac{1}{Z_2(s)}} \frac{V_{in}(s)}{Z_1(s)}$$

$$= \frac{1}{Y_1(s) + Y_2(s)} \frac{V_{in}(s)}{Z_1(s)}$$

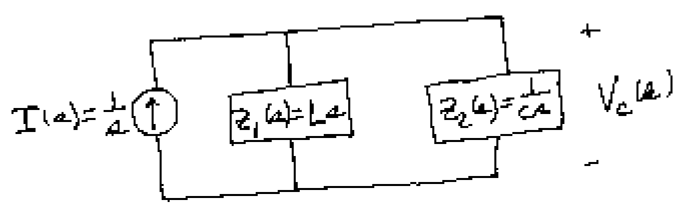
This is readily seen to correspond to the circuit shown below:



The equivalence of these two circuit structures is often convenient in facilitating application of voltage or current division.

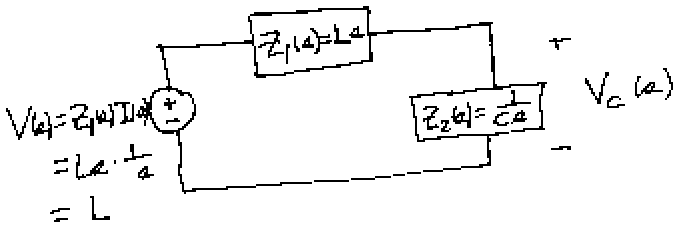
Example Consider the circuit shown below, where initial conditions are zero, the input current is a unit-step signal, and the output signal is the voltage across the capacitor:

Converting this to the transform equivalent circuit gives



A source transformation leads to another equivalent circuit in the transform domain, where the current source is replaced by a transform voltage source

$$V(s) = Z_1(s)I(s) = Ls\frac{1}{s} = L$$



Next the voltage divider relation gives

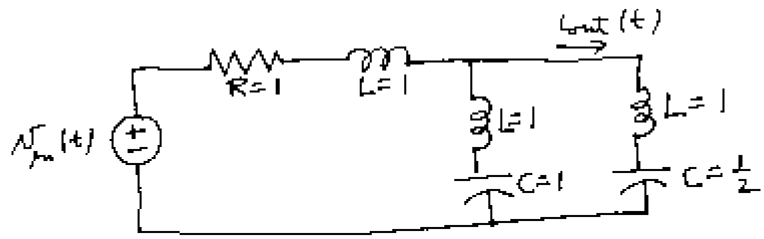
$$V_C(s) = \frac{Z_2(s)}{Z_1(s)+Z_2(s)}\,V(s) = \frac{1/(Cs)}{Ls+1/(Cs)}\,L$$

$$= \frac{1/C}{s^2+1/(LC)} = \sqrt{L/C}\;\frac{1/\sqrt{LC}}{s^2+1/(LC)}$$

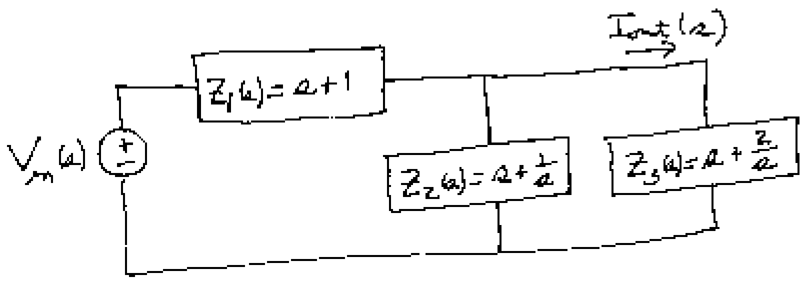
and table lookup gives the inverse Laplace transform

$$v_C(t) = \sqrt{L/C}\;\sin(\tfrac{1}{\sqrt{LC}}\,t)\,u(t)$$

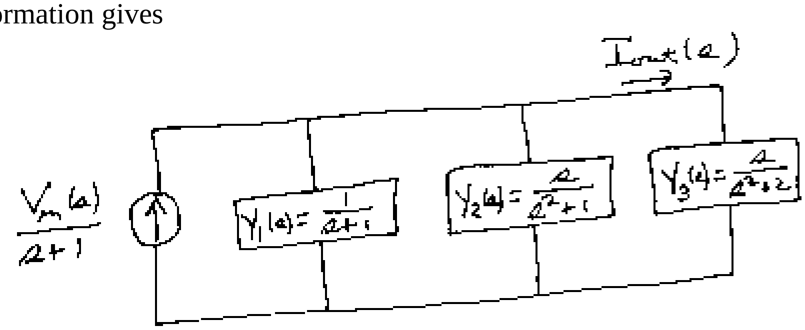
Example These basic approaches are quite efficient, even for reasonably complicated circuits. Consider the case below, where the objective is to compute the transfer function $I_{out}(s)/V_{in}(s)$:



Converting to the transform equivalent circuit gives



where we have chosen impedance labels for the various portions of the overall circuit. Next, a source transformation gives



Finally, by current division,

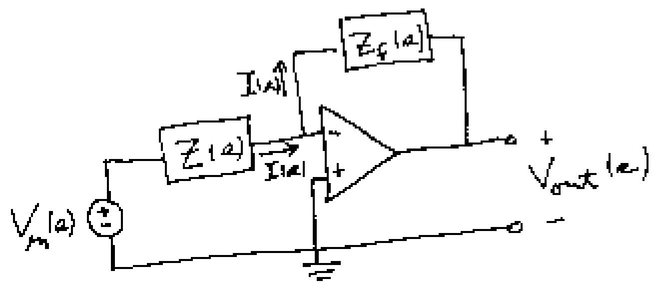
$$I_{out}(s) = \frac{Y_3(s)}{Y_1(s)+Y_2(s)+Y_3(s)} \frac{V_{in}(s)}{s+1}$$

$$= \frac{\frac{s}{s^2+2}}{\frac{1}{s+1}+\frac{s}{s^2+1}+\frac{s}{s+2}} \frac{V_{in}(s)}{s+1}$$

$$= \frac{s(s^2+1)}{3s^4+2s^3+6s^2+3s+2} V_{in}(s)$$

The notion of a transform equivalent circuit can be used in other settings that are not restricted to RLC circuits. We illustrate this by considering a circuit involving an ideal operational amplifier.

Example Beginning directly in the Laplace domain, consider the ideal op amp with impedances $Z(s)$ and $Z_f(s)$ as shown:



In order to compute the voltage transfer function , $V_{out}(s)/V_{in}(s)$, we use the *virtual short* property of the ideal op amp to conclude that
$V_{out}(s) = -Z_f(s)I(s)$
and also

$$V_{in}(s) = Z(s)I(s)$$

Thus it is easy to see that

$$\frac{V_{out}(s)}{V_{in}(s)} = -\frac{Z_f(s)}{Z(s)}$$

For example, if we choose a capacitor in the feedback path and a resistor in the input path, then $Z_f(s) = 1/(Cs)$ and $Z(s) = R$. This gives

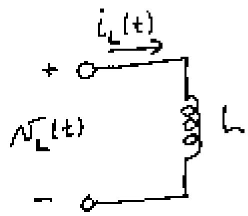
$$\frac{V_{out}(s)}{V_{in}(s)} = -\frac{1/(RC)}{s}$$

and in the time domain we recognize that the circuit is a running integrator.

12.2 Circuits with Nonzero Initial Conditions

For circuit elements with nonzero initial stored energy, that is, nonzero initial conditions, we can develop Laplace transform equivalent circuits that represent the initial conditions as voltage or current sources.

For an inductor,



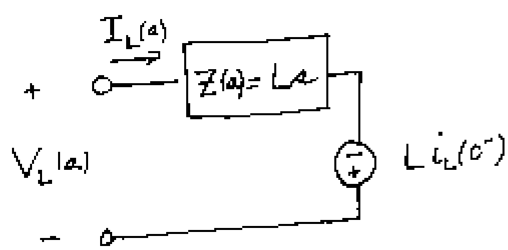
with initial current $i_L(0^-)$ in the indicated direction, the voltage-current relation in the time domain remains

$$v_L(t) = L\frac{di_L(t)}{dt}, \quad t \geq 0$$

However, the unilateral Laplace transform differentiation property, taking account of the initial condition, yields

$$\begin{aligned} V_L(s) &= L[sI_L(s) - i_L(0^-)] \\ &= LsI_L(s) - Li_L(0^-) \end{aligned}$$

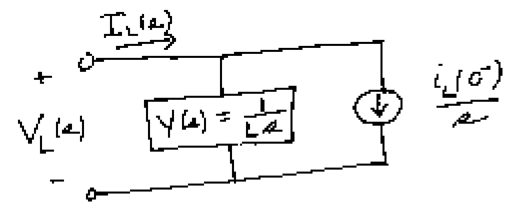
This corresponds to the transform equivalent circuit shown below, where the initial condition term is represented as a voltage source with appropriate polarity:



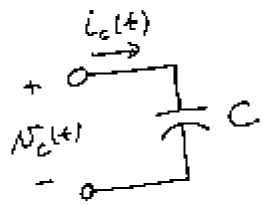
Of course, an alternate approach is to write

$$I_L(s) = \frac{1}{Ls}V_L(s) + \frac{i_L(0^-)}{s}$$

This leads to an admittance version of the transform equivalent, where the initial condition is represented as a current source with appropriate polarity:



Similar calculations for a capacitor are almost apparent. With an initial voltage $v_C(0^-)$, the capacitor with polarity as marked



is described by

$$C\,\dot{v}_C(t) = i_C(t)\,, \quad t \geq 0$$

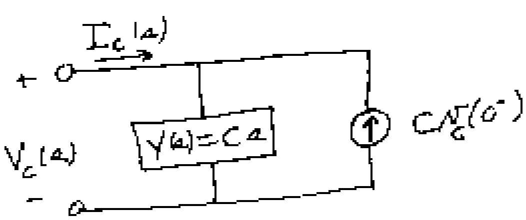
The Laplace transform differentiation property gives

$$C[sV_C(s) - v_C(0^-)] = I_C(s)$$

or

$$I_C(s) = CsV_C(s) - Cv_C(0^-)$$

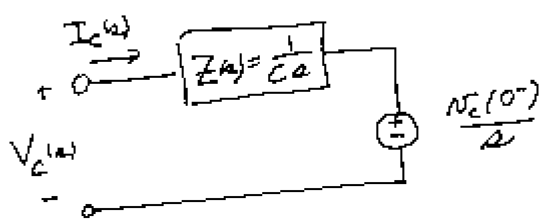
This corresponds to the transform equivalent circuit shown below, where the initial condition is represented by a current source.



An alternative is to write

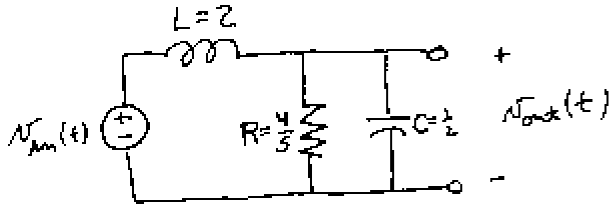
$$V_C(s) = \frac{1}{Cs}\,I_C(s) + \frac{v_C(0^-)}{s}$$

which corresponds to the circuit

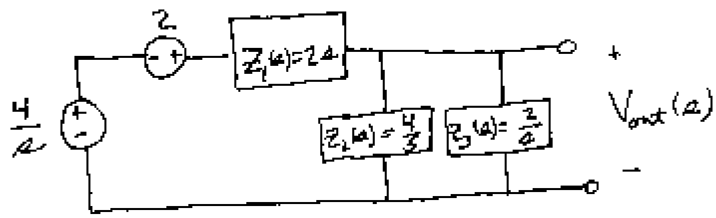


where a voltage source accounts for the initial condition.

Example Consider the circuit



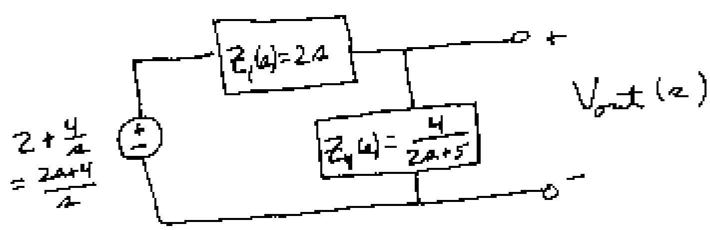
where the input voltage is $v_{in}(t) = 4u(t)$, the initial current in the inductor is $i_L(0^-) = 1$, and the initial voltage on the capacitor is zero. To compute the output, $v_{out}(t)$, we first sketch the transform equivalent circuit:



The two voltage sources can be combined, and impedances in parallel can be combined according to

$$Z_4(s) = \frac{Z_2(s)Z_3(s)}{Z_2(s) + Z_3(s)} = \frac{4}{2s+5}$$

This gives the equivalent circuit shown below



Now a straightforward voltage-divider calculation gives $V_{out}(s)$:

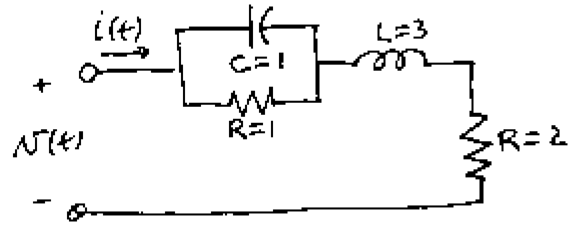
$$V_{out}(s) = \frac{\frac{4}{2s+5}}{2s + \frac{4}{2s+5}} \frac{2s+4}{s} = \frac{2s+4}{s^3 + \frac{5}{2}s^2 + s} = \frac{2}{s(s+\frac{1}{2})}$$

and partial fraction expansion leads to

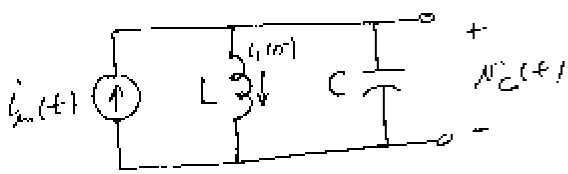
$$v_{out}(t) = 4u(t) - 4e^{-\frac{1}{2}t}u(t)$$

Exercises

1. Compute the impedance $Z(s)$ for the circuit shown below.



2. For the circuit shown below, with L=1 and C=1, suppose the input current is $i_{in}(t) = 2u(t)$, the initial current in the inductor is $i_L(0^-) = 1$ in the direction shown, and the initial voltage on the capacitor is zero. Compute the voltage output, $v_C(t)$.



3. Consider the electrical circuit shown below where the input voltage is $v_{in}(t) = u(t)$ and the initial conditions are $i_L(0^-) = 1$, $v_C(0^-) = 0$. Compute the current through the resistor, $i_R(t)$ for $t \geq 0$.