8

Frequency domain design

We now consider the effect of controllers in the frequency domain.

8.1 Loop shaping

When we refer to loop shaping, we are really considering the shape of the Bode magnitude plot. Specifically, consider the loop gain:

$$L(s) = P(s)C(s).$$

What do we expect this loop gain to look like?

8.2 Tracking

If we consider the need to track reference signals, $r(t)$, we look at the tracking error:

$$e(t) = r(t) - y(t).$$

In the Laplace transform domain, we get:

$$\begin{aligned} E(s) &= R(s) - Y(s) \\ &= R(s) - \frac{P(s)C(s)}{1 + P(s)C(s)} R(s) \\ &= \underbrace{\frac{1}{1 + P(s)C(s)}}_{S(s)} R(s). \end{aligned}$$

The function $S(s)$ is known as the sensitivity function. Note that, in the frequency domain:

$$|E(j\omega)| = |S(j\omega)||R(j\omega)|.$$

Thus, for the tracking error to be small at any given frequency ω, either the magnitude of the sensitivity function, or the magnitude of the reference function has to be small at that frequency. One way of achieving this is for the sensitivity function to be zero at that particular frequency. This is what having an internal model achieves.

Example 13. Suppose that we want to track a sinusoid:

$$r(t) = \sin \omega_0 t$$

If the system is internally stable, the steady-state tracking error is given by

$$e(t) = |S(j\omega_0)| \sin(\omega_0 t - \angle S(j\omega_0)) + \text{transients}.$$

We learned earlier that by including an internal model of the disturbance into the controller we can achieve zero steady-state error. In particular, we set

$$C(s) = \frac{1}{s^2 + \omega_0^2} C'(s).$$

Then, provided that $P(j\omega_0) \neq 0$,

$$\begin{aligned} S(s)|_{s=j\omega_0} &= \left. \frac{1}{1 + P(s)C(s)} \right|_{s=j\omega_0} \\ &= \left. \frac{s^2 + \omega_0^2}{s^2 + \omega_0^2 + P(s)C'(s)} \right|_{s=j\omega_0} \\ &= 0. \end{aligned}$$

Thus,

$$\lim_{t \to \infty} e(t) = |S(j\omega_0)| \sin(\omega_0 t - \angle S(j\omega_0)) = 0.$$

□

In the example above, we achieve zero error by ensuring that the sensitivity function is exactly zero at the frequency of interest. In practice, this is not always possible, as the following example demonstrates.

Example 14. Suppose that we want to track a square wave:

$$r(t) = \text{sign}(\sin \omega_0 t)$$

where

$$\text{sign}(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases}$$

To see that this is not a signal that we could track with zero error, even in steady-state, we write the signal in terms of a Fourier series:

$$r(t) = \sum_{n=1,3,5,\ldots} r_n(t),$$

where

$$r_n(t) = \frac{4}{n\pi}\sin(n\omega_0 t).$$

From linearity, the tracking error due to the external disturbance $r(t)$ is the sum of the tracking errors due to each of the constituent components $r_n(t)$; i.e.

$$e(t) = \sum_{n=1,3,5,\ldots} e_n(t).$$

As above, we can make any one of these errors go to zero by ensuring that there is an internal model of the signal $r_n(t)$ in the controller. However, to get the error to go to zero completely would require that each of the $e_n(t)$ go to zero, and this means that the controller would have to have internal models of the fundamental frequency, ω_0, and all the harmonics $n\omega_0$, $n = 3, 5, \ldots$. Clearly, this is not practical. □

Though perfect tracking may not be possible, we can settle for tracking to a particular frequency-dependent requirement. As an example, we can require that

$$|S(j\omega)| \leq \begin{cases} \epsilon \ll 1, & \text{if } |\omega| < \Omega, \\ M, & \text{if } |\omega| \geq \Omega. \end{cases}$$

We can selecte $\Omega = m\omega_0$ for some odd integer m.

Example 15. We continue the previous example, where we are tracking the square wave. In the frequency-domain,

$$|E_n(j\omega)| = |S(j\omega)||R_n(j\omega)|.$$

Because the input is a sinusoid, the steady-state error is also a sinusoid given by

$$e_n(t) = \frac{4}{n\pi}|S(jn\omega_0)|\sin(n\omega_0 t - \angle(S(jn\omega_0)))$$

We focus on the magnitude of the sinusoid $e_n(t)$.

$$\begin{aligned} \frac{4}{n\pi}|S(jn\omega_0)| &\leq \frac{4\epsilon}{n\pi}, \qquad (\text{for } 1 \leq n \leq m.) \\ &\leq \frac{4\epsilon}{\pi}, \end{aligned}$$

where the last inequality came from taking the fundamental component ($n = 1$).

For higher frequencies:

$$\begin{aligned} \frac{4}{n\pi}|S(jn\omega_0)| &\leq \frac{4M}{n\pi}, \qquad (\text{for } n > m) \\ &\leq \frac{4M}{m\pi}, \qquad (\text{for } n > m). \end{aligned}$$

Once again, the last inequality came from taking the largest magnitude component in this frequency range, which is $n = m$. Thus, the magnitude of every signal $e_n(t)$is given by:

$$\frac{4}{n\pi}|S(jn\omega_0)| \leq \frac{4}{\pi}\min\left\{\epsilon, \frac{M}{m}\right\}.$$

Thus, by choosing

$$m \geq \frac{M}{\epsilon}$$

we ensure that the magnitude of the sinusoid $e_n(t)$:

$$|e_n(t)| \leq \frac{4}{n\pi}|S(jn\omega_0)| \leq \frac{4}{\pi}\epsilon, \qquad n = 1, 3, \ldots.$$

□

It is tempting to try to make both bounds ϵ and M be arbitrarily small and the frequency range Ω arbitrarily large. In Section 8.3 we will see why this is not always possible or even desirable.

8.2.1 Ensuring bounds on $S(s)$ through the loop gain

Let's suppose that, to achieve our tracking error requirements, we need

$$|S(j\omega)| < \epsilon.$$

In terms of the loop gain, this is equivalent to

$$|1 + L(j\omega)| > \frac{1}{\epsilon}.$$

We want to write this requirement in terms of the actual loop gain ($|L(j\omega)|$). To do this we make use of some properties of complex numbers. For any two complex numbers z_1 and z_2

$$\max\{|z_1| - |z_2|, |z_2| - |z_1|\} \leq |z_1 + z_2| \tag{8.1}$$

In particular, note that of the two terms in the first inequality, only one can be positive. Let $z_1 = 1$ and $z_2 = L(j\omega)$. Then, assuming that $|L(j\omega)| > 1$:

$$|L(j\omega)| - 1 \leq |1 + L(j\omega)|.$$

Thus, requiring that

$$|L(j\omega)| \geq 1 + \frac{1}{\epsilon} = \frac{1+\epsilon}{\epsilon} \tag{8.2}$$

ensures that

$$|1 + L(j\omega)| \geq |L(j\omega)| - 1 \geq \frac{1}{\epsilon},$$

and, hence:

$$|S(j\omega)| < \epsilon.$$

In practice,

$$\epsilon \ll 1 \Rightarrow |L(j\omega)| \geq \frac{1+\epsilon}{\epsilon} \approx \frac{1}{\epsilon} \gg 1.$$

is sufficient. Note that this also ensures that our assumption that $|L(j\omega)| > 1$ is correct.

8.2.2 A tracking example

8.3 Robustness

Here we are interested in the stability of the closed-loop system under the assumption that the controller has been designed based on a model of the true system that may not be completely accurate. We consider two plants. The first is denoted by $P(s)$ and represents the nominal model of the system to be controlled. Because $P(s)$ is the model that is available to the designer, the controller $C(s)$ is chosen so as to stabilize the closed-loop system with $P(s)$. The second plant is denoted by $P_\Delta(s)$ and is assumed to represent the true system to be controlled. However, because of uncertainties or other restrictions, the true plant is not available to the designer. Hence, the robust design problem is to design a controller $C(s)$ based on $P(s)$ that works with the unknown $P_\Delta(s)$.

Of course, if $P_\Delta(s)$ were completely unknown, it would be impossible to carry out this task. However, the underlying assumption is that $P(s)$ is a reasonable model for $P_\Delta(s)$. The relationship between the two is assumed to satisfy

$$P_\Delta(s) = P(s)\,(1 + \Delta(s))\,.$$

The function $\Delta(s)$ can be thought of as relative error in the model. If $|\Delta(j\omega)|$ is small, then the model is accurate at that frequency. Otherwise, the error is large.

Example 16. Suppose that the true plant is known, but includes a delay:

$$P_\Delta(s) = \frac{1}{(0.01s+1)^2} e^{-0.1s}.$$

In the nominal model, we ignore the delay:

$$P(s) = \frac{1}{(0.01s+1)^2}.$$

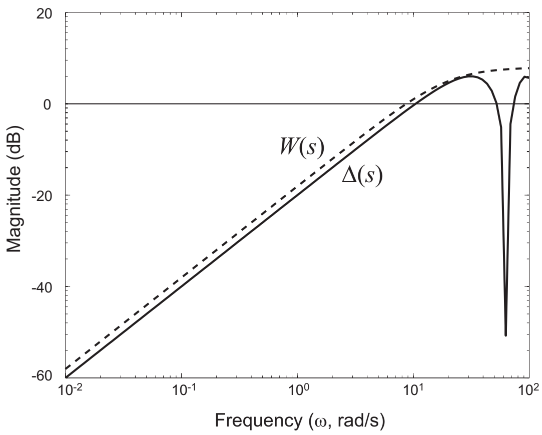

Fig. 8.1. Bode magnitude plot of $\Delta(s) = e^{-0.1s} - 1$ (solid line) and of the function $W(s) = 0.125s/(s/20 + 1)$ (dashed line).

The modeling error is

$$\Delta(s) = \frac{P_\Delta(s)}{P(s)} - 1 = e^{-0.1s} - 1.$$

In the frequency domain:

$$\Delta(j\omega) = \cos(0.1\omega) - 1 - j\sin(0.1\omega)$$

and

$$|\Delta(j\omega)|^2 = (\cos(0.1\omega) - 1)^2 + \sin^2(0.1\omega) = 2(1 - \cos(0.1\omega)).$$

This shows that the magnitude of the modeling error ranges between 0 and $2 \approx 6\,\mathrm{dB}$. In particular, as can be seen from the Bode plot (Fig. 8.1), the error is bounded by $|W(j\omega)|$ where

$$W(s) = \frac{0.125s}{s/20 + 1}.$$

□

Because we design controllers to stabilize $P(s)$, but actually implement them on $P_\Delta(s)$, it is natural to ask whether the real closed-loop system will continue to be stable. Systems for which this property is true are said to be *robust*.

To analyze this question, we consider the closed-loop system depicted in Fig. 8.2A and its equivalent form in Fig. 8.2B, with signals $V(s)$ and $W(s)$ which are the input and output signals into $\Delta(s)$. Note that

$$V(s) = -P(s)C(s)[V(s) + W(s)].$$

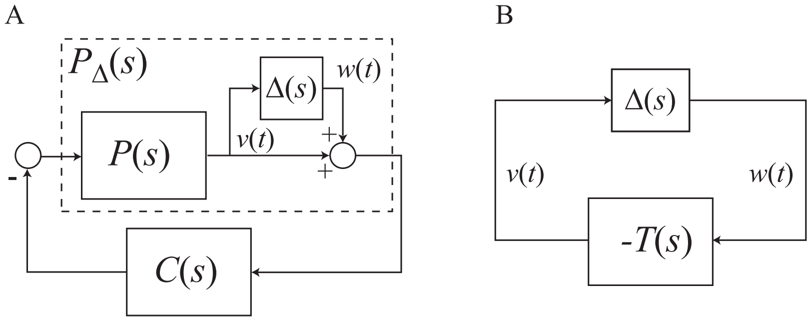

Fig. 8.2. A. Closed-loop system with the real plant $P_\Delta(s) = P(s)(1 + \Delta(s))$. The signals $v(t)$ and $w(t)$ represent the input and outputs to the uncertainty block $\Delta(s)$. B. Equivalent representation of the closed-loop system.

Rearranging this equation leads to

$$V(s) = -\underbrace{\frac{P(s)C(s)}{1 + P(s)C(s)}}_{T(s)} W(s).$$

The function $T(s)$ is known as the complementary sensitivity function. The following result tells us when the real closed-loop system is stable.

Theorem 4. *Suppose that $P_\Delta(s) = P(s)[1 + \Delta(s)]$ with $\Delta(s)$ stable. If the closed-loop system with $P(s)$ and $C(s)$ is internally stable, then the closed-loop system with $P_\Delta(s)$ and $C(s)$ is stable if*

$$|T(j\omega)||\Delta(j\omega)| < 1, \qquad \forall\omega. \tag{8.3}$$

Proof. The proof follows from an application of the Nyquist stability criterion. Note that $\Delta(s)$ and $T(s)$ are both stable (the first by assumption, the second from the assumption about internal stability of the nominal system). Hence, the Nyquist stability criterion states that the closed-loop system with loop $T(s)\Delta(s)$ is stable if the loop gain $T(j\omega)\Delta(j\omega)$ does not go through the -1 point and encircles it exactly zero times in a counterclockwise fashion. The condition given by Eq. 8.3, however, guarantees that the Nyquist plot of the loop gain remains inside the unit circle (see Fig. 8.3). Hence, it can not touch, much less encircle, the -1 point. Hence the system is stable.

We can now use this theorem to determine whether a system is robust to unmodeled dynamics.

Example 17. We return to our previous example. We found that

$$|\Delta(j\omega)| \leq 2, \qquad \forall\omega.$$

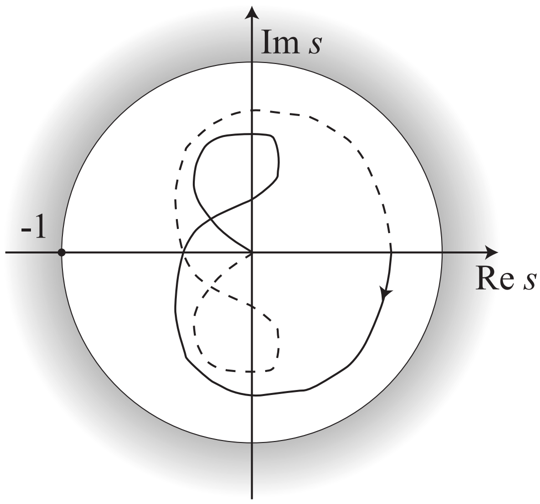

Fig. 8.3. A. Closed-loop system with the real plant $P_\Delta(s) = P(s)(1 + \Delta(s))$. The signals $v(t)$ and $w(t)$ represent the input and outputs to the uncertainty block $\Delta(s)$. B. Equivalent representation of the closed-loop system.

Hence, if the nominal closed-loop system satisfies

$$|T(j\omega)| < \tfrac{1}{2}, \qquad \forall\omega,$$

then the closed-loop system will continue to be stable even under the presence of the delay. $\square$

It follows from Eq. 8.3 that to make the system as robust as possible requires that $|T(j\omega)|$ be small.

8.3.1 Ensuring bounds on $T(s)$ through the loop gain

We showed above that the condition on the sensitivity function for tracking could be turned into a sufficient condition on the loop gain. We now do the same for the complementary sensitivity function.

Suppose that, to achieve our robustness requirements, we need

$$|T(j\omega)| < \epsilon.$$

In terms of the loop gain, this is equivalent to

$$0 < \frac{|L(j\omega)|}{|1 + L(j\omega)|} < \epsilon.$$

Once again, we turn to the complex number inequalities Eq. 8.1, by setting $z_1 = 1$ and $z_2 = L(j\omega)$. However, this time we assume that $|L(j\omega)| < 1$. Then

$$0 < 1 - |L(j\omega)| \leq |1 + L(j\omega)|,$$

which implies that

$$\frac{1}{|1 + L(j\omega)|} \leq \frac{1}{1 - |L(j\omega)|}.$$

Multiplying by $|L(j\omega)|$ yields:

$$|T(j\omega)| = \frac{|L(j\omega)|}{|1 + L(j\omega)|} \leq \frac{|L(j\omega)|}{1 - |L(j\omega)|}.$$

Thus, requiring that

$$|L(j\omega)| < \frac{\epsilon}{1 + \epsilon} \tag{8.4}$$

ensures that

$$|T(j\omega)| < \epsilon.$$

Again, in practice,

$$\epsilon \ll 1 \Rightarrow |L(j\omega)| \leq \frac{\epsilon}{1 + \epsilon} \approx \epsilon \ll 1.$$

is sufficient. Note that this also ensures that our assumption that $|L(j\omega)| < 1$ is correct.

8.4 Performance-robustness tradeoffs

We have seen that, for good tracking, we require that the sensitivity function be small. On the other hand, good robustness is ensured when the complementary sensitivity function is small. This presents a problem. In fact, the "complementary" in the name for $T(s)$ refers to the fact that

$$S(s) + T(s) = 1.$$

It should be clear, then, that both can not be small. In fact, we saw this in the conflicting requirements of the loop gain. Good tracking is satisfied by

$$|L(j\omega)| \geq \frac{1 + \epsilon}{\epsilon} \Rightarrow |S(j\omega)| \leq \epsilon$$

but this leads to large loop gains. In contrast, robustness is achieved whenever

$$|L(j\omega)| \leq \frac{\epsilon}{1 + \epsilon} \Rightarrow |T(j\omega)| \leq \epsilon,$$

but this requires small loop gains.

We note that

$$S(j\omega) + T(j\omega) = 1,$$

has to be satisfied for all frequencies. In particular, if $|S(j\omega)| \approx \epsilon$, then $|T(j\omega)| \approx 1$. Alternatively, if $|T(j\omega)| \approx \epsilon$, then $|S(j\omega)| \approx 1$. It follows that $|S(j\omega)|$ and $|T(j\omega)|$ can not both be small at any one frequency. However, both can be small at different frequencies. In practice, our requirements for small $|S(j\omega)|$ and $|T(j\omega)|$ do not usually appear in the same frequency range.

For tracking, we require that

$$|S(j\omega)||R(j\omega)|$$

be small. However, most signals that are tracked in practice have most of their energy in the lower frequencies (take a look at Example 15). Hence,

$$|R(j\omega)| \approx \epsilon_l, \qquad |\omega| > \Omega_l$$

for some ϵ and Ω. Thus, it is only necessary to make sure that

$$|S(j\omega)| \leq \epsilon_l, \qquad |\omega| \leq \Omega_l,$$

and

$$|S(j\omega)| \approx 1, \qquad |\omega| > \Omega_l.$$

Similarly, for robustness we require that

$$|T(j\omega)||\Delta(j\omega)| < 1.$$

However, most models tend to be quite accurate at low frequencies (see Example 16). Thus

$$|\Delta(j\omega)| \approx \epsilon_h, \qquad |\omega| < \Omega_h.$$

Hence, to assure robustness it is sufficient to ensure that

$$|T(j\omega)| \leq \epsilon_h, \qquad |\omega| \geq \Omega_h,$$

and

$$|T(j\omega)| \approx 1, \qquad |\omega| < \Omega_h,$$

Thus, we have two sets of requirements on the loop gain:

$$|L(j\omega)| \geq \frac{1+\epsilon_l}{\epsilon_l}, \qquad \forall|\omega| < \Omega_l,$$

and

$$|L(j\omega)| \leq \frac{\epsilon_h}{1+\epsilon_h}, \qquad \forall|\omega| > \Omega_h,$$

over non-overlapping frequency ranges: $\Omega_l < \Omega_h$; see Fig. 8.4.

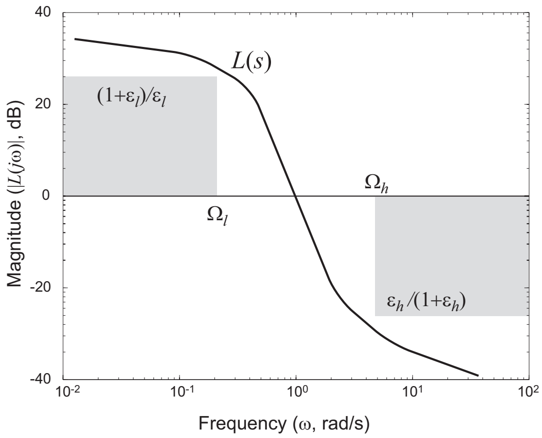

Fig. 8.4.

8.5 Bode's gain phase relationship

The previous section argued that requirements for both performance (tracking) and robustness can be expressed in terms of frequency-dependent bounds on the magnitude Bode plot of the loop gain. However, designs are not of any use unless the closed-loop system is stable. In this section we analyze the effect of loop gains on stability.

This is somewhat of a tricky aspect. In particular, we know from the Nyquist stability criterion that stability is tied to both the magnitude and the phase of the loop transfer function. However, in looking at the loop gain, we have only been concentrating on the magnitude of the loop transfer function. Can we, by looking only at the loop gain, say anything about the phase? The answer, in general, is no. As a simple example, consider the following two loop gains:

$$L_1(s) = \frac{1}{s+2}, \qquad L_2(s) = \frac{1}{s-2}.$$

The magnitude of both functions is

$$|L_1(j\omega)| = |L_2(j\omega)| = \frac{1}{\sqrt{\omega^2+4}}.$$

However, the phases are different:

$$\angle L_1(j\omega) = -\tan^{-1}(\omega/2), \qquad \angle L(_2 j\omega) = -\pi + \tan^{-1}(\omega/2).$$

In fact, the two loop gains give rise to quite different closed-loop systems. The characteristic polynomial for $L_1(s)$ is:

$$\varDelta_1(s) = 1 + (s+2) = s+3$$

which has roots in the open-left-hand plane, and hence the closed-loop system is internally stable. However, the characteristic polynomial for $L_2(s)$ is:

$$\Delta_2(s) = 1 + (s - 2) = s - 1.$$

This has a root at $+1$, and hence the closed-loop system is unstable.

As we can not conclude anything about the phase from the magnitude plot for a general transfer function, we can ask whether there are classes of systems for which we can conclude something about the phase from the magnitude. It turns out that there is. A *minimum phase* transfer function is one in which all poles and zeros are in the closed, left half-plane. For these functions, the following relationship tells you the relationship between phase and magnitude.

Theorem 5 (Bode, 1945). *Suppose that $L(s)$ is stable and minimum phase, and assume that $L(0) > 0$. Then, for every angle ω_0:*

$$\angle L(j\omega_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{d}\ln|L(je^{\nu})|}{\mathrm{d}\nu} \ln\coth\frac{|\nu|}{2}\,\mathrm{d}\nu, \tag{8.5}$$

where $\nu = \ln(\omega/\omega_0)$.

Clearly, the relationship between magnitude and phase is quite complicated and no one would ever suggest that you might compute that integral to obtain phase from the magnitude. However, the integral relationship shows that the magnitude and phase of the loop transfer function, $L(s)$, can not be set arbitrarily. Moreover, we can learn a few things about the relationsip from this function.

First, note that it is not the magnitude itself that determines the phase, but the *slope* of the magnitude function. Hence, $kG(s)$ and $G(s)$ have the same phase for any $k > 0$.

Second, consider a system with transfer function

$$L(s) = \frac{1}{s^n}.$$

The magnitude is

$$|L(j\omega)| = \frac{1}{|\omega|^n} \Rightarrow |L(je^{\nu})| = \frac{1}{|\omega/\omega_0|^n}.$$

and

$$\frac{\mathrm{d}\ln|L(je^{\nu})|}{\mathrm{d}\nu} = \frac{1}{|L(je^{\nu})|}\frac{d|L(je^{\nu})|}{\mathrm{d}e^{\nu}}\frac{\mathrm{d}e^{\nu}}{\mathrm{d}\nu}.$$

The first term is:

$$\frac{1}{|L(je^{\nu})|} = |je^{\nu}|^n = e^{n\nu}.$$

The second is:

$$\frac{\mathrm{d}|L(je^{\nu})|}{\mathrm{d}e^{\nu}} = -ne^{-\nu(n+1)}.$$

The third is: $\mathrm{d}e^{\nu}/\mathrm{d}\nu = e^{\nu}$. Putting all three together yields:

$$\frac{\mathrm{d}\ln|L(e^{\nu})|}{\mathrm{d}\nu} = -e^{n\nu} \times ne^{-\nu(n+1)} \times e^{\nu} = -n.$$

Hence, the first term in the integrand is independent of frequency and can be pulled outside the integral:

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{d}\ln|L(je^{\nu})|}{\mathrm{d}\nu} \ln\coth\frac{|\nu|}{2}\,\mathrm{d}\nu = -\frac{n}{\pi}\int_{-\infty}^{\infty} \ln\coth\frac{|\nu|}{2}\,\mathrm{d}\nu.$$

The term:

$$\ln\coth\frac{|\nu|}{2} = \ln\left|\frac{\omega+\omega_0}{\omega-\omega_0}\right|, \tag{8.6}$$

and

$$\int_{-\infty}^{\infty} \ln\coth\frac{|\nu|}{2}\,\mathrm{d}\nu = \frac{\pi^2}{2}.$$

Hence,

$$\angle L(j\omega) = -\frac{n\pi}{2}.$$

In summary, and in terms of the usual Bode plot nomenclature, for this class of systems the phase and magnitude are related by

$$|L(j\omega)| = -20n\,\mathrm{dB/decade} \iff \angle L(j\omega) = -90^\circ n.$$

To understand the relationship in the general case, it is instructive to plot the integrand in Eq. 8.6; see Fig. 8.5. It can be thought of as a weighting function, by which the phase at angle ω_0, though in theory depends on the slope of the magnitude function at all frequencies, it depends most heavily on the frequencies near ω_0. The following example illustrates this point.

Example 18. Consider the transfer function

$$L(s) = \frac{1}{s(s+10)}.$$

The magnitude function is

$$|L(j\omega)| = \frac{1}{|\omega|\sqrt{100+\omega^2}}.$$

At low frequencies ($|\omega| < 1$), the magnitude $|L(j\omega)| \approx 1/|\omega|$, and hence the slope is -20 dB/decade. The phase is approximately -90°. For high frequencies ($|\omega| > 100$), $|L(j\omega)| \approx 1/|\omega|^2$. The slope is approximately -40 dB/decade and, correspondingly, the phase is approximately -180°. □

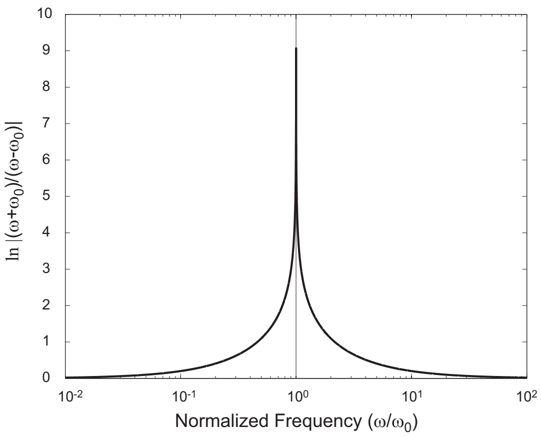

Fig. 8.5. Weighting function in Bode's gain-phase integral. It is centered around $\nu = 0$ which corresponds to $\omega = \omega_0$. The area in the range $\nu \in [-\ln 10, \ln 10]$ (equivalent to $\omega \in [\omega_0/10, 10\omega_0]$) is approximately 92% of the total area. This means that 92% of the phase at the angle ω_0 is determined by the slope of the magnitude in that two decade frequency range.

8.5.1 Loop-shaping requirement around the cross-over frequency

From the previous section, we can estimate the phase of the loop gain from the slope of the magnitude plot — at least for minimum phase transfer functions. What should this phase be? If the loop gain is minimum phase, there are no unstable open-loop poles, and hence the Nyquist plot should not encircle the -1 point. This means that, near the cross-over frequency ω_{gc}, the phase should not be near $-180°$. Thus, from Theorem 5, the slope of the magnitude should be around -20 dB/decade. This places a limit on the size of the drop in magnitude from $1/\epsilon_l$ at ω_l to ϵ_h at ω_h.

8.6 A loop-shaping design example

Consider the plant:

$$P_\Delta(s) = \frac{1}{s(s+1)} e^{-sT}$$

where $T = 0.1$ represents a delay. We will design a controller based on the model

$$P(s) = \frac{1}{s(s+1)}.$$

We would like to meet three specs:

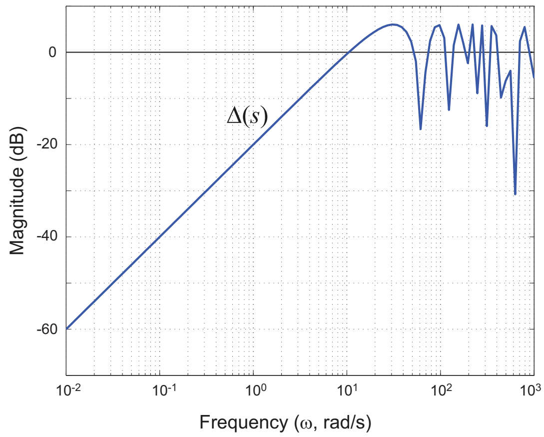

Fig. 8.6. Frequency-dependent magnitude of the relative error in the model. For frequencies less than 10 rad/s, the error is less than 0 dB. Higher frequencies lead to errors greater than 0 dB. This means that the gain of $T(s)$ must be below 0 dB at those higher frequencies.

1. Tracking: we want to track sinusoids of frequencies $\omega \in [0, \Omega_l]$ with error less than 5%. The exact frequency range Ω_l is to be determined, but the bigger the better.
2. We would like to achieve large phase margin.
3. We want to be sure that the controller, when implemented on $P_\Delta(s)$, will provide a stable closed-loop system.

Solution.

We computed the uncertainty model

$$\Delta(s) = \frac{P_\Delta(s)}{P(s)} - 1 = e^{-sT} - 1.$$

earlier. By plotting the magnitude of this function (see Fig. 8.6), we see that the uncertainty in the model is significant after 10 rad/s. By then, the loop gain should be smaller than 0 dB; thus, the bandwidth should not exceed 10 rad/s.

We now determine what a reasonable frequency Ω_l over which we can track sinusoids. To ensure that the sinusoids are tracked with error less than 5%, we need:

$$|S(j\omega)| \leq 0.05 \approx -26\,\text{dB}$$

This can be guaranteed provided that

$$|L(j\omega)| \geq \frac{1}{0.05} + 1 = 21 \approx 26.4\,\text{dB}.$$

However, the delay imposes a bandwidth requirement of less than 10 rad/s. Let us pick 5 rad/s. If the loop gain is decreasing at -20 dB/decade, the phase margin will be good (because, from Theorem 5, the phase should be approximately $-90°$). Starting at 0 dB, this will mean that the highest frequency for which a gain of -20 dB/decade will still be above 26.4 dB is $26.4/20 \approx 1.32$ decades below 5 rad/s. This works out to be:

$$\Omega_l = 5 \times 10^{-1.32} \approx 0.23\,\mathrm{rad/s}.$$

We now design a controller to achieve these two goals. We first use a loop transfer function that resembles an integrator as much as possible. However, because the plant has relative degree two (two poles, no zeros), the loop gain must also be of the same form — otherwise, the controller turns out to be improper. Thus, we look for loop gains of the form:

$$L_1(s) = \frac{k}{s(s+p)} \iff C_1(s) = \frac{L_1(s)}{P(s)} = \frac{k(s+1)}{s+p}.$$

The integrator in this loop gain ensures that steps will be tracked with zero steady-state error. This was not stated as a specific requirement, but it is a desirable feature of most systems. Moreover, because the open-loop has a pole there, we can take advantage of it.

The presence of the integrator means that, at low frequencies, the phase is around $-90°$. The pole p should not be too close to the gain crossover frequency. Otherwise, it starts pushing the phase closer to $-180°$, and the phase margin is then small. If we put this pole one decade away, then the phase should be reasonable. Thus, we pick:

$$L_1(s) = \frac{k}{s(s+100)}$$

It only remains to pick the gain k. The gain cross-over frequency is given by the solution of:

$$|L_1(j\omega_{\mathrm{gc}})| = 1 \iff 1 = \frac{k}{\omega_{\mathrm{gc}}\sqrt{\omega_{\mathrm{gc}}^2 + 10,000}}.$$

However, since we selected the pole of 100 to be above the gain cross-over frequency, this is approximately:

$$\frac{k}{\omega_{\mathrm{gc}}\sqrt{\omega_{\mathrm{gc}}^2 + 10,000}} \approx \frac{k}{100\omega_{\mathrm{gc}}}$$

which means that $\omega_{\mathrm{gc}} \approx k/100$. Suppose that we select $k = 800$. Thus:

$$L_1(s) = \frac{800}{s(s+100)} \Longrightarrow C(s) = \frac{L(s)}{P(s)} = \frac{800(s+1)}{(s+100)}$$

To make sure that the system is stable, we check whether

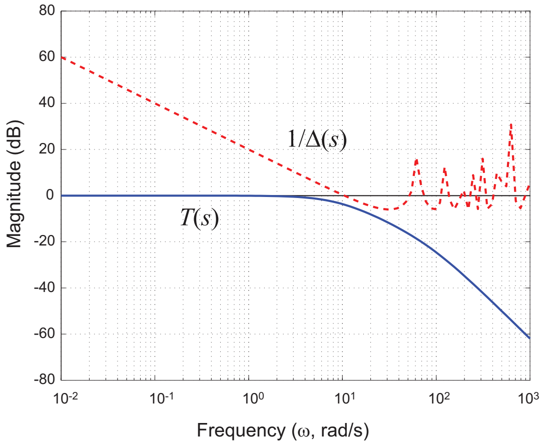

Fig. 8.7. Robust stability can be guaranteed if $|T(j\omega)| < 1/|\Delta(j\omega)|$ for all frequencies. In deciBels, this corresponds to the requirement that $|T(j\omega)|$ be below $1/|\Delta(j\omega)|$ Shown is the frequency-dependent magnitude of $T(s)$ (solid line) and of $1/\Delta(s)$ (dotted). Clearly, the robust stability requirement is met.

$$|T(j\omega)| < \frac{1}{|\Delta(j\omega)|}.$$

This is plotted in Fig. 8.7 which shows that the requirement on robust stability is met. We can check the tracking requirement by plotting the sensitivity function; see Fig. 8.8.

From Fig. 8.8, it is clear that whenever $|\omega| < 0.4\,\mathrm{rad/s}$ the sensitivity is less than about -26 rad/s. Thus, $\Omega_l \approx 0.4\,\mathrm{rad/s}$. This loop gain gives a phase margin of about $85°$.

It is reasonable to ask whether we can do better. After all, the choices of the pole at $100\,\mathrm{rad/s}$ and the gain $k = 800$ were selected somewhat arbitrarily. We could increase the gain at lower frequencies. From Fig. 8.8, this would increase Ω_l. For example:

$$L_2(s) = L_1(s) \times \frac{k_2(s+1)}{s+0.01}.$$

The fact that, in the new component ($\frac{k_2(s+1)}{s+0.01}$), the pole (0.01 rad/s) appears at lower frequency than the zero (1 rad/s) means that we are increasing the gain at lower frequencies. We choose the gain k_2 to make sure that we leave the cross-over frequency intact. Thus, k_2 is chosen so that the gain of the $\frac{k_2(s+1)}{s+0.01}$ is one at the gain cross-over frequency ($\omega_{\mathrm{gc}} = 8$):

$$k_2 = \sqrt{\frac{\omega_{\mathrm{gc}}^2 + 0.01^2}{\omega_{\mathrm{gc}}^2 + 1^2}} = \sqrt{\frac{8^2 + 0.01^2}{8^2 + 1^2}} \approx 1.$$

Thus

$$L_2(s) = \frac{800(s+1)}{s(s+0.01)(s+100)} \Longrightarrow C_2(s) = \frac{800(s+1)^2}{(s+0.01)(s+100)}$$

From the Bode plot of the two loop transfer functions (Fig. 8.9), we see that the high frequency behavior is largely unaffected. The low frequency magnitude has been increased.

We now double-check the robust stability requirement and note that it is still being satisfied; see (Fig. 8.10). In fact, we see that the two complementary sensitivity functions are nearly identical. This should not be surprising. The complementary sensitivity function $T(s) = L(s)/(1 + L(s))$. At lower frequencies, the two loop transfer functions differ, but they both have magnitudes signficantly greater than one. Thus, $|T(j\omega)| \approx 1$ for both. At higher frequencies, the loop gain $|T(j\omega)| \approx |L_1(j\omega)| \approx |L_2(j\omega)|$.

The sensitivity function is shown in Fig. 8.11. The range of frequencies where $|S(j\omega)| < -26\,\mathrm{dB}$ has gone from about 0.4 to 0.7 rad/s. The phase margin has decreased to 78°. At this point one could try to increase the complexity of the controller in the same way that we went from L_1 to L_2. Clearly, the phase margin is still quite high; thus, some of this could be sacrificed to increase the range of frequencies over which the tracking requirement is met. However, because the increase in Ω_l was minor, it is probably not worth the added complexity.

Though the design seems quite good, it is always a good idea to do one last check: make sure that the nominal closed-loop system (that is, with $P(s)$ and $C(s)$) is stable. We do this by checking the roots of the characteristic polynomial:

$$s(s+0.01)(s+100) + 800(s+1) \approx (s+91.3)(s+7.51)(s+1.17).$$

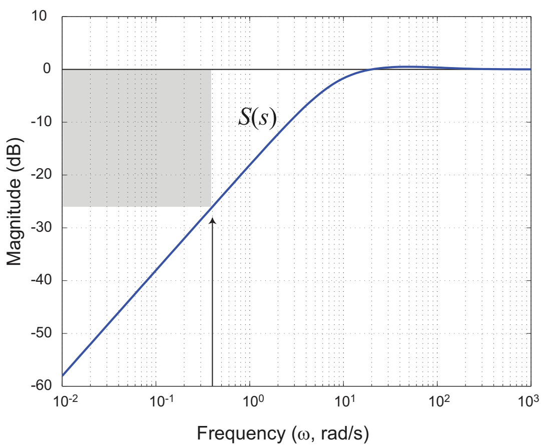

Fig. 8.8. Bode magnitude plot of the sensitivity function. This function lies below $-26\,\mathrm{dB}$ for frequencies less than $\Omega_l \approx 0.4\,\mathrm{rad/s}$. Thus, any sinusoid with frequency less than Ω_l will be tracked with less 5% error at steady-state.

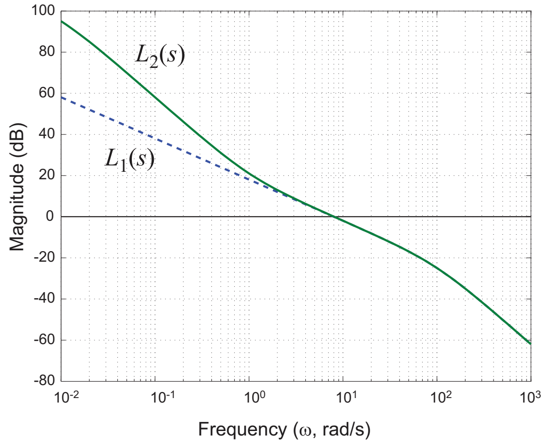

Fig. 8.9. Bode plot for the loop transfer functions $L_1(s)$ (dotted line) and $L_2(s)$ (solid). The magnitude is unaffected for frequencies higher than about 1 rad/s. The phase, which is not shown, would lag for frequencies approximately between about 0.001 and 10 rad/s.

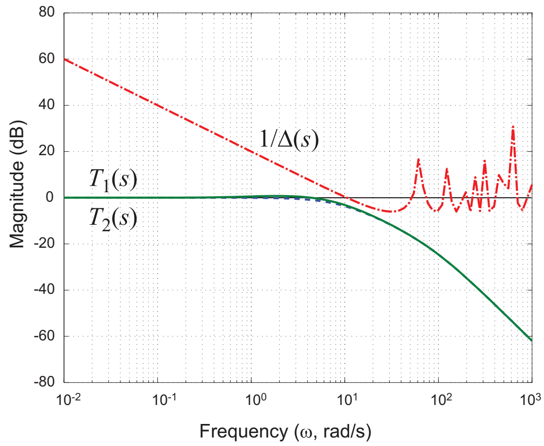

Fig. 8.10. Magnitude Bode plots for the two complementary sensitivity functions $T_1(s)$ (dotted line) and $T_2(s)$ (solid) from $L_1(s)$ and $L_2(s)$, respectively. Both are below the plot of $1/|\Delta(j\omega)|$ (dashed-dot line) guaranteeing robust stability.

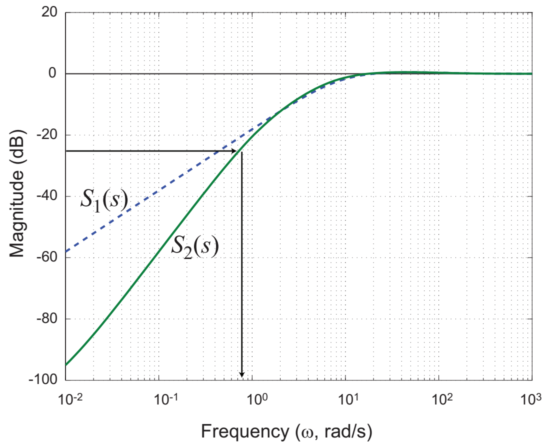

Fig. 8.11. Magnitude Bode plots for the two sensitivity functions $S_1(s)$ (dotted line) and $S_2(s)$ (solid) from $L_1(s)$ and $L_2(s)$, respectively. Note that the increased low-frequency gain of $L_2(s)$ lowers the sensitivity function at these frequencies. This results in better tracking of sinusoids.

Note that all the roots are in the left half-plane and so the nominal system is stable .

8.7 Effects of non-minimum phase plants