You have seen that the design of a controller can be broekn down into the following two parts:

- 1. Designing a state feedback regulator u = -Kx; and
- 2. Building a state observer.

You can design controllers where the closed-loop poles are placed at any desired location. At this point, you might want to ask the following question. Is there some *K* that is better than others? This question leads you into the realm of *optimal control theory*. That is, designing controllers which optimize some desirable characteristic. The first, and also best known, problem that has been considered is the *linear quadratic regulator problem*. Its rigorous derivation is somewhat tricky and best left to a graduate course in the area. We can certainly do much better than the presentation in the book, however, so these notes should help you get started.

# 7.1 The Problem

We begin with the single-input, single-output system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t),$$

with initial condition  $x(0) = x_0$ . We will assume that the system is controllable and observable.

Our goal is to minimize a combination of the output and input values:

$$J(u) = \int_0^\infty y^2(t) + \rho u^2(t) \,\mathrm{d}t$$

We are already starting to be somewhat non-rigorous (mathematicians would hate us). Note that I have written an improper integral without worrying whether it is well

defined. We will assume that any control input we look for is such that  $u(t) \rightarrow 0$ and  $y(t) \rightarrow 0$  sufficiently fast so as to make the integral finite. The parameter  $\rho > 0$ is used to weigh the two different goals of the integrand. Large  $\rho$  penalizes large input signals (this serves as a means of preventing saturations, for example). Small  $\rho$ makes the output smaller.

At this point, we do not know much, but let us assume that the state-feedback input u(t) = -Kx(t) is used. Note that there is no *a priori* reason to expect that the optimal input would be a state-feedback.

If u(t) = -Kx(t), then the integrand can be written as:

$$y^{2}(t) + \rho u^{2}(t) = x^{T}(t)[C^{T}C + \rho K^{T}K]x(t)$$

Also, since

$$\dot{x}(t) = (A - BK)x(t) \quad \Rightarrow \quad x(t) = e^{(A - BK)t}x_0$$

where A - BK needs to be stable (otherwise the integrand would probably blow up!) Hence,

$$J(Fx) = x_0^T \left[ \int_0^\infty e^{(A - BK)^T t} [C^T C + \rho K^T K] e^{(A - BK)t} dt \right] x_0$$
  
=  $x_0^T X x_0$ ,

where we have defined the integral to be the symmetric matrix X. It follows that the cost function is quadratic in the initial condition  $x_0$ .

Let us perform some matrix manipulations on X. These are slightly unmotivated at this point, but you will later see how they arise.

$$[A - BK]^{T}X + X[A - BK]$$

$$= \int_{0}^{\infty} (A - BK)^{T} e^{(A - BK)^{T}t} [C^{T}C + \rho K^{T}K] e^{(A - BK)t} dt$$

$$+ \int_{0}^{\infty} e^{(A - BK)^{T}t} [C^{T}C + \rho K^{T}K] e^{(A - BK)t} (A - BK) dt$$

$$= \int_{0}^{\infty} \frac{d}{dt} \left( e^{(A - BK)^{T}t} [C^{T}C + \rho K^{T}K] e^{(A - BK)t} \right) dt$$

$$= e^{(A - BK)^{T}t} [C^{T}C + \rho K^{T}K] e^{(A - BK)t} \Big|_{t=0}^{\infty}$$
(7.1)
$$= 0 - [C^{T}C + \rho K^{T}K]$$
(7.2)

where we first used the fundamental theorem of calculus in Eq. 7.1, and then the (assumption) that  $e^{(A+BF)t} \rightarrow 0$  in Eq. 7.2.

It follows that

$$[A - BK]^{T}X + X[A - BK] + C^{T}C + \rho K^{T}K = 0$$

but this can also be written as:

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$$A^{T}X + XA - \frac{1}{\rho}XBB^{T}X + C^{T}C + \rho(K - \frac{1}{\rho}B^{T}X)^{T}(K - \frac{1}{\rho}B^{T}X) = 0 \quad (7.3)$$

This is an equation that relates the feedback matrix K to the cost function  $J(Fx) = x_0^T X x_0$ . No notion of optimality has been used so far. Note, however, that X must be a positive matrix, because the integrand is always positive.

At this point, we *choose* a particular pair (K, X) related by this equation. In particular, we are going to make the last quadratic term 0; that is  $K = \frac{1}{a}B^T P$  and

$$A^{T}P + PA - \frac{1}{\rho}PBB^{T}P + C^{T}C = 0$$
(7.4)

We've used a different letter (P instead of X) to remind you that this is the cost of a particular choice of K. We will also use the following characterization of this equation

$$[A - \frac{1}{\rho}BB^TX]^TP + P[A - \frac{1}{\rho}BB^T] + \frac{1}{\rho}PBB^TP + C^TC = 0$$

This is just Eq. 7.3 with the specific choices of X and K.

Let's start checking a few things for this choice of K. First of all, we will see that the resultant closed-loop system is stable. This will allay fears that the integrand is blowing up.

**Lemma 2.** With P the positive semi-definite solution to Eq. 7.4, the matrix  $A_K := A - BK = A - \frac{1}{\rho}BB^TP$  is stable.

*Proof.* We prove this by contradiction. Suppose that  $A_K$  is *not* stable; that is, there exists an eigenvalue  $\lambda$  and eigenvector v such that

$$A_K v = \lambda v \tag{7.5}$$

with Re  $\lambda \ge 0$ . Note that, in general, eigenvalues and eigenvectors are complex numbers. Consider the *Hermitian* transpose of Eq. 7.5. A Hermitian transpose, denoted by  $X^H$  is just the regular transpose with complex conjugates. That is:

$$X^H = \overline{[X^T]} = (\bar{X})^T.$$

Note that, if the matrix or vector X is real, then  $X^H = X^T$ . It follows that the Hermitian transpose of Eq. 7.5 is

$$v^H A_K^T = \bar{\lambda} v^H \tag{7.6}$$

Now, premultiplying equation Eq. 7.4 by  $v^H$  and postmultiplying by v, yields:

$$v^{H} \left( A_{K}^{T} P + PA_{K} + \frac{1}{\rho} PBB^{T} P + C^{T} C \right) v$$
$$= v^{H} P v (\lambda + \bar{\lambda}) + v \left( \rho PBB^{T} P + C^{T} C \right) v$$

Consider the individual elements of this equation. First of all, since P is a positive semidefinite matrix,  $v^H P v \ge 0$ , for any vector v. Also  $(\lambda + \overline{\lambda}) = 2 \operatorname{Re} \lambda \ge 0$ .

Finally, note that  $v^H C^T C v = ||Cv||^2 \ge 0$  and  $\rho v (PBB^T P) v = \rho ||B^T P v||^2 \ge 0$ . It follows that all the elements are non-negative. Hence, the only way they can sum to zero is if they are *all* zero. But then Cv = 0 and

$$Av = A_K v + \frac{1}{\rho} B B^T P v = \lambda v$$

But these two facts contradict observability, since:

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} v = \begin{bmatrix} Cv \\ CAv \\ \vdots \\ CA^{n-1}v \end{bmatrix} = \begin{bmatrix} Cv \\ \lambda Cv \\ \vdots \\ \lambda^{n-1}Cv \end{bmatrix} = 0_{n \times 1}$$

It must be that  $A_K$  is stable.  $\Box$ 

OK, so far so good. Our system is stable. Now we need to show that this is, in fact, the best we can do. We begin by considering the following function of time

$$V(t) = x^T(t)Px(t)$$

Computing its derivative

$$\begin{split} \dot{V}(t) &= \dot{x}^T(t) P x(t) + x^T(t) P \dot{x}(t) \\ &= \left[ A x(t) + B u(t) \right]^T P x(t) + x^T(t) P \left[ A x(t) + B u(t) \right] \\ &= \left[ x^T(t) \ u^T(t) \right] \begin{bmatrix} A^T P + P A \ P B \\ B^T P \ 0 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}. \end{split}$$

Now, integrate left and right-hand sides from 0 to  $\infty$ . This leads to

$$V(\infty) - V(0) = \int_0^\infty \left[ x^T(t) \ u^T(t) \right] \begin{bmatrix} A^T P + PA \ PB \\ B^T P \ 0 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt$$
$$= 0 - x_0^T P x_0$$

where all we have assumed is that the control u causes the state x to go to zero. Thus:

$$\int_0^\infty \left[ x^T(t) \ u^T(t) \right] \begin{bmatrix} A^T P + PA \ PB \\ B^T P \ 0 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt + x_0^T P x_0 = 0$$
(7.7)

Let's now work with J:

$$J(u) = \int_{0}^{\infty} \begin{bmatrix} x^{T}(t) \ u^{T}(t) \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & \rho I \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt$$
$$= \int_{0}^{\infty} \begin{bmatrix} x^{T}(t) \ u^{T}(t) \end{bmatrix} \begin{bmatrix} A^{T}P + PA + Q \ PB \\ B^{T}P \quad \rho I \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt + x_{0}^{T}Px_{0} \quad (7.8)$$
$$= \int_{0}^{\infty} \begin{bmatrix} x^{T}(t) \ u^{T}(t) \end{bmatrix} \begin{bmatrix} \frac{1}{\rho} PBB^{T}P \ PB \\ B^{T}P \quad \rho I \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt + x_{0}^{T}Px_{0} \quad (7.9)$$

$$= \int_{0}^{\infty} \left[ \frac{1}{x} x^{T}(t) PB u^{T}(t) \right] \begin{bmatrix} B^{T}P & \rho I \\ D^{T}P & \rho I \end{bmatrix} \begin{bmatrix} u(t) \end{bmatrix} dt + x_{0}^{T} Px_{0}$$
(7.10)  
$$= \int_{0}^{\infty} \left[ \frac{1}{x} x^{T}(t) PB u^{T}(t) \right] \begin{bmatrix} B^{T}P & \rho I \\ D^{T}P & \rho I \end{bmatrix} \begin{bmatrix} x(t) \\ D^{T}P & \rho I \end{bmatrix}$$

$$= \int_{0}^{\infty} \begin{bmatrix} \frac{1}{\rho} x^{T}(t) PB u^{T}(t) \end{bmatrix} \begin{bmatrix} B^{T}P' \rho I \end{bmatrix} \begin{bmatrix} u(t) \\ u(t) \end{bmatrix} dt + x_{0}^{t} Px_{0}$$
(7.10)  
$$\int_{0}^{\infty} \begin{bmatrix} 1 u^{T}(t) PB u^{T}(t) \end{bmatrix} \begin{bmatrix} \rho I \rho I \end{bmatrix} \begin{bmatrix} \frac{1}{\rho} B^{T} Px(t) \end{bmatrix} dt + u^{T} Px_{0}$$
(7.11)

$$= \int_{0}^{\infty} \left[ \frac{1}{\rho} x^{T}(t) PB u^{T}(t) \right] \left[ \frac{\rho I}{\rho I} \frac{\rho I}{\rho I} \right] \left[ \frac{\rho D}{u(t)} \frac{1}{u(t)} dt + x_{0}^{T} Px_{0} \quad (7.11) \\ = \rho \int_{0}^{\infty} \left( \frac{1}{\rho} x^{T}(t) PB + u^{T}(t) \right) \left( \frac{1}{\rho} B^{T} Px(t) + u(t) \right) dt + x_{0}^{T} Px_{0} \\ = \rho \int_{0}^{\infty} \left[ \frac{1}{\mu} u(t) + \frac{1}{\mu} B^{T} Px(t) \right] dt + x_{0}^{T} Px_{0} \quad (7.11)$$

$$= \rho \int_0^\infty \|u(t) + \frac{1}{\rho} B^T P x(t)\|^2 \, \mathrm{d}t + x_0^T P x_0$$

where: in line Eq. 7.8 we have added Eq. 7.7; in line Eq. 7.9 we used Eq. 7.4 in the (1,1) term of the block matrix; in line Eq. 7.10 we transferred the terms  $\frac{1}{\rho}PB$  to the  $x^{T}$  term; in line Eq. 7.11 we transferred the terms  $\frac{1}{\rho}B^{T}P$  to the x term.

Note that the cost function J(u) divides into two parts. The first — the integral — is always non-negative. The second term is independent of u. It follows that the best you can do is to set the first term equal to zero; but this is accomplished by the choice

$$\frac{1}{\rho}B^T P x(t) + u(t) = 0 \quad \Rightarrow \quad u(t) = -K x(t)$$

as we claimed.

At this point, we should really show that a solution to Eq. 7.4 does exists. I'm going to skip this part of the theory as I think that it takes us a bit far afield. Instead, it would be more instructive to look at an example.

Example 11. Consider the first order system:

$$\dot{x}(t) = ax(t) + u(t), \qquad y = x$$

and suppose that we want to minimize  $\int_0^\infty y^2(t) + u^2(t) dt$ . We will not specify a. The theory says that the optimal control input u(t) = -Kx(t) where K = p and p satisfies

$$0 = 2ap - p^2 + 1 \quad \Rightarrow \quad p = a \pm \sqrt{a^2 + 1}$$

Clearly, there are two solutions. We want to take the one that has  $p \ge 0$ , and this is  $p = a + \sqrt{a^2 + 1}$ . The control input is then u = -kx where

$$k = p = a + \sqrt{a^2 + 1}$$

placing the closed-loop pole at  $a - k = -\sqrt{a^2 + 1}$ .  $\Box$ 

## 7.2 Optimal Observers

We will not develop the theory too much, other than by using our intuition. Remember that we formulated the observer problem by noting that

(C, A) is observable  $\iff (A^T, C^T)$  is controllable

The problem of finding an observer gain L such that A+LC is stable is equivalent to finding the gain  $K^T$  such that  $A^T - C^T K^T$  is stable. We might be tempted to define an optimal observer gain by considering the optimal control gain for the problem with  $A^T$  and  $C^T$ . This leads to another Algebraic Riccati Equation:

$$AQ + QA^T - \frac{1}{\rho}QC^TCQ + BB^T = 0$$
(7.12)

and observer gain

$$L = \frac{1}{\rho} Q C^T. \tag{7.13}$$

Note that we have just taken Eq. 7.4, and replaced A with  $A^T$ ; B with  $C^T$  and C with  $B^T$ .

It can be shown that this is indeed an optimal observation problem for a suitably defined problem.

In particular, consider the system

$$\dot{x}(t) = Ax(t) + Bw(t)$$
$$y(t) = Cx(t) + v(t).$$

The two external signals: w(t) and v(t) are stochastic signals; that is, these are signals where at any point in time their value is a random variable.

We make the following assumptions about the probability distributions. First, both signals are zero mean random processes. That is, for any t:

$$\mathcal{E}w(t) = 0,$$
 and  $\mathcal{E}v(t) = 0,$ 

where  $\mathcal{E}$  denotes mathematical expectation. Because both signals are zero mean, and the system is linear, then the state and output will both be zero mean processes. Furthermore, we assume that

$$\mathcal{E}w(t)w(\tau) = \delta(t-\tau), \quad \mathcal{E}v(t)v(\tau) = \rho\delta(t-\tau), \text{ and } \mathcal{E}v(t)w(\tau) = 0,$$

for any t and  $\tau$ . The first two are equivalent to assuming that the two disturbances are white noise processes. The third assumes that there is no correlation between them.

Now, we build a observer for this system:

$$\tilde{x}(t) = A\tilde{x}(t) + L[y(t) - C\tilde{x}(t)].$$

Notice that the observer does not include components from the external disturbances as these are assumed to be unmeasurable. The estimation error is  $e(t) = x(t) - \tilde{x}(t)$  and obeys the differential equation:

$$\dot{e}(t) = (A - LC)e(t) + Bw(t) + Lv(t).$$

Thus:

$$e(t) = e^{A_L t} e_0 + \int_0^t e^{A_L(t-\sigma)} \left( Bw(\sigma) + Lv(\sigma) \right) d\sigma,$$
(7.14)

where we have defined  $A_L = A - LC$  to keep the notation somewhat more manageable.

At this point, we can ask what is a reasonable cost function for the estimator. Clearly, we want e(t) small. However, because the external disturbances are zero mean, the integral will be as well. A more meaningful measure of the estimation error, is then

$$\lim_{t \to \infty} \mathcal{E} \| e(t) \|^2.$$

Note that, because we are looking at the variance of the estimate as  $t \uparrow \infty$ , then the first term in Eq. 7.14 will go to zero, provided that  $A_L$  is stable. Thus,

$$\lim_{t\to\infty} \mathcal{E} \| e(t) \|^2 = \lim_{t\to\infty} \mathcal{E} \| \tilde{e}(t) \|^2$$

where

$$\tilde{e}(t) = \int_0^t e^{A_L(t-\sigma)} \left( Bw(\sigma) + Lv(\sigma) \right) d\sigma.$$

Now we evaluate this, but we do so indirectly. Recall from linear algebra that

$$\|\tilde{e}(t)\|^2 = [\tilde{e}(t)]^T \tilde{e}(t) = \operatorname{trace}\left(\tilde{e}(t)[\tilde{e}(t)]^T\right).$$

Moreover,

$$\tilde{e}(t)[\tilde{e}(t)]^T = \int_0^t \int_0^t e^{A_L[t-\sigma_1]} [Bw(\sigma_1) + Lv(\sigma_1)] [Bw(\sigma_2) + Lv(\sigma_2)]^T e^{A_L^T[t-\sigma_2]} d\sigma_1 d\sigma_2.$$

Note that the term in the center of the integral equals:

$$Bw(\sigma_1)w(\sigma_2)B^T + Bw(\sigma_1)v(\sigma_2)L^T + Lv(\sigma_1)w(\sigma_2)B^T + Lv(\sigma_1)v(\sigma_2)L^T$$
(7.15)

When we take the mathematical expectation, because both the trace operation and multiplication by the matrix exponential are linear, we get

$$\mathcal{E}\|\tilde{e}(t)\|^{2} = \mathcal{E}\operatorname{trace}\left(\tilde{e}(t)[\tilde{e}(t)]^{T}\right)$$
  
= trace  $\int_{0}^{t} \int_{0}^{t} e^{A_{L}[t-\sigma_{1}]} \mathcal{E}[\cdots] e^{A_{L}^{T}[t-\sigma_{2}]} d\sigma_{1} d\sigma_{2}.$ 

where the term omitted is Eq. 7.15. Thus

$$\begin{aligned} \mathcal{E}[\cdots] = & B \, \mathcal{E}[w(\sigma_1)w(\sigma_2)]B^T + B \, \mathcal{E}[w(\sigma_1)v(\sigma_2)]L^T \\ & + L \, \mathcal{E}[v(\sigma_1)w(\sigma_2)]B^T + L \, \mathcal{E}[v(\sigma_1)v(\sigma_2)]L^T \\ = & [BB^T + \rho L L^T]\delta(\sigma_1 - \sigma_2). \end{aligned}$$

Replacing this into the cost function, and integrating with respect to  $\sigma_2$ , leads to:

$$\mathcal{E}\|\tilde{e}(t)\|^{2} = \operatorname{trace} \int_{0}^{t} \int_{0}^{t} e^{A_{L}[t-\sigma_{1}]} [BB^{T} + \rho LL^{T}] \delta(\sigma_{1} - \sigma_{2}) e^{A_{L}^{T}[t-\sigma_{2}]} d\sigma_{1} d\sigma_{2}$$
$$= \operatorname{trace} \int_{0}^{t} e^{A_{L}[t-\sigma_{1}]} [BB^{T} + \rho LL^{T}] e^{A_{L}^{T}[t-\sigma_{1}]} d\sigma_{1}.$$

Now, define  $\tau = t - \sigma_1$ :

$$\mathcal{E}\|\tilde{e}(t)\|^2 = \operatorname{trace} \int_0^t e^{A_L \tau} [BB^T + \rho L L^T] e^{A_L^T \tau} \, \mathrm{d}\tau.$$

We are almost done. Note that, defining

$$\hat{A} = A^T, \quad \hat{B} = C^T, \quad \hat{C} = B^T, \quad \text{and} \quad \hat{K} = L^T$$

then

$$\lim_{t \to \infty} \mathcal{E} \|\tilde{e}(t)\|^2 = \operatorname{trace} \int_0^\infty e^{\hat{A}_K^T t} [\hat{C}^T \hat{C} + \rho \hat{K}^T \hat{K}] e^{\hat{A}_K t} dt.$$

This is the integral that appears in the LQR problem (Eq. ??) under the assumption that state-feedback control is used u = -Kx:

$$J(u = -\hat{K}x) = x_0^T \int_0^\infty e^{\hat{A}_K^T \tau} [\hat{C}^T \hat{C} + \rho \hat{K}^T \hat{K}] e^{\hat{A}_K \tau} dt x_0.$$

Thus, the controller  $\hat{K}$  that minimizes this J(u) also minimizes

 $\lim_{t\to\infty} \mathcal{E} \|\tilde{e}(t)\|^2.$ 

What is this controller? It is given by

$$\hat{K} = -\frac{1}{\rho}\hat{B}^T\hat{P}$$

where P comes from the solution of the equation

$$0 = \hat{A}^T \hat{P} + \hat{P} \hat{A} + \hat{C}^T \hat{C} - \frac{1}{\rho} \hat{P} \hat{B} \hat{B}^T \hat{P}$$

In terms of the original data, this is simply Eq. 7.12 and Eq. 7.13 with  $Q = \hat{P}$ .

*Example 12.* Need a good one!  $\Box$ 

# 7.3 Optimal controllers

We now consider the problem of designing an optimal controller C(s) for the stateequation Eq. ??. This is the same problem that we considered in Section 7.1 except that we assume that we do not have access to the state, so the state feedback control of Section 7.1 can not be used. The really nice (and surprising) fact is that the optimal controller is the following:

$$\dot{\hat{x}} = A\hat{x} + Bu - L(y - C\hat{x})$$
$$u = Fx$$

where  $F = -\frac{1}{\rho}B^T P$  and  $L = -QC^T$  and P and Q are the solutions to the AREs Eq. 7.4 and Eq. 7.12. That is, the optimal controller is the observer controller where the two components are optimal for the separate problems. This fact is now known as the *separation principle*.

OK, let's begin. I am going to define a signal

$$v = u - Fx$$

and consider the new system, by replacing the input u with the new input v:

$$\dot{x} = (A + BF)x + Bi$$
$$y = Cx$$

The state equation has as solution

$$\begin{aligned} x(t) &= e^{(A+BF)t} x_0 + \int_0^\infty e^{(A+BF)(t-\tau)} Bv(\tau) \, d\tau \\ &= e^{(A+BF)t} x_0 + (g_v * v)(t) \end{aligned}$$

where  $g_v(t) = e^{(A+BF)t}B$  is the impulse response of the system which transfers the input v to the state x. This system has transfer function

$$G_v(s) = (sI - A - BF)^{-1}B$$

I can also write down the first term in the solution for x(t) as a convolution of  $g_x(t) = e^{(A+BF)t}$  with the signal  $x_0\delta(t)$ . Note that

$$G_x(s) = (sI - A - BF)^{-1}$$

It follows that

$$X(s) = G_x(s)x_0 + G_v(s)V(s)$$

Now, we want to minimize

$$\int_0^\infty y^2(t) + u^2(t) \, \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^\infty |Y(j\omega)|^2 + |U(j\omega)|^2 \mathrm{d}\omega$$

By Parceval's theorem. Note that:

$$|Y(j\omega)|^2 + |U(j\omega)|^2 = \| \begin{bmatrix} Y(j\omega) \\ U(j\omega) \end{bmatrix} \|^2$$

Also,

$$\begin{bmatrix} Y(s) \\ U(s) \end{bmatrix} = \begin{bmatrix} C \\ F \end{bmatrix} X(s) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} V(s)$$

$$= \begin{bmatrix} C \\ F \end{bmatrix} G_x(s)x_0 + \begin{bmatrix} CG_v(s) \\ 1 + FG_v(s) \end{bmatrix} V(s)$$

$$= H_x(s)x_0 + H_v(s)V(s)$$

Hence

$$|Y(j\omega)|^{2} + |U(j\omega)|^{2} = x_{0}^{T}H_{x}(j\omega)^{H}H_{x}(j\omega)x_{0} + |v(j\omega)|^{2}H_{v}(j\omega)^{H}H_{v}(j\omega) + 2x_{0}^{T}H_{x}(j\omega)^{H}H_{v}(j\omega)$$