6

Observers

The previous chapter showed you how to obtain a state-feedback controller that places the closed-loop eigenvalues is desired locations. Of course, this requires that your control system have access to all the states. In practice, this will not always be possible. Here we show how, using the output and a model of the system, an estimate of the states can be obtained. A system that does this successfully is known as an *observer*.

6.1 Full-order observer design

We will assume that the system is given by

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = x_0 \\ y(t) &= Cx(t) + Du(t).\end{aligned}$$

We want to design a system that takes y and u as inputs and produces an estimate, $\tilde{x}$, of the state.

If the estimate $\tilde{x}$ is going to behave like the real state, the dynamics of the estimated state should be similar to that of the real system. Thus, we define a model:

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + Bu(t).$$

It follows that the difference between the real and estimated states

$$e(t) = x(t) - \tilde{x}(t),$$

which we call the state error, satisfies the differential equation

$$\begin{aligned}\dot{e}(t) &= \dot{x}(t) - \dot{\tilde{x}}(t) \\ &= Ax(t) + Bu(t) - A\tilde{x}(t) - Bu(t) \\ &= Ae(t)\end{aligned}$$

with initial condition:

$$e(0) = x(0) - \tilde{x}(0) = e_0.$$

Note that,

$$e(t) = \mathrm{e}^{At} e_0,$$

which, if A is aymptotically stable, implies that $e(t) \to 0$.

What happens if A is *not* stable? Let us modify the estimated state's equation:

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + Bu(t) + L\big(y(t) - C\tilde{x}(t) - Du(t)\big),$$

where $L \in \mathbf{R}^{n\times 1}$ is a "free" matrix to be chosen.

The term inside the brackets:

$$y(t) - C\tilde{x}(t) - Du(t),$$

can be thought of as an estimate of the output of the system, because

$$e_y = y - C\tilde{x} - Du = (Cx + Du) - (C\tilde{x} + Du) = C(x - \tilde{x}) = Ce.$$

It follows that, if the estimate e goes to zero, so does $e_y(t)$. However, the converse is not true, in general. It is possible for $e_y(t)$ to go to zero without $e(t)$. However, if the system is observable, then this reverse implication does hold.

Once again, we compute the dynamic equation for $e(t)$:

$$\begin{aligned}
\dot{e}(t) &= \dot{x}(t) - \dot{\tilde{x}}(t) \\
&= Ax(t) + Bu(t) - \Big(A\tilde{x}(t) + Bu(t) + L\big(y(t) - C\tilde{x}(t) - Du(t)\big)\Big) \\
&= (A - LC)e(t)
\end{aligned}$$

The error decays to zero if we can find an L such that $A - LC$ is asymptotically stable. Note that the eigenvalues of $A - LC$ are the same as those of

$$(A - LC)^T = A^T - C^T L^T.$$

Define

$$\bar{A} = A^T, \quad \bar{B} = C^T, \quad \bar{K} = L^T$$

Then, the problem reduces to finding a $\bar{K}$ such that $\bar{A} - \bar{B}\bar{K}$ has the desired eigenvalues.

This is the same problem that we considered in looking for a state-feedback control input. Specifically, we know that if the pair $(\bar{A}, \bar{B})$ is controllable, then the eigenvalues of $\bar{A} - \bar{B}\bar{K}$ can be set arbitrarily. Recall that $(\bar{A}, \bar{B})$ is controllable if and only if the controllability matrix

$$\bar{\mathcal{C}} = \big[\bar{B} \cdots \bar{A}^{n-1}\bar{B}\big]$$

is of full rank. Replacing $\bar{A} = A^T$ and $\bar{B} = C^T$ we get:

$$
\begin{aligned}
\bar{\mathcal{C}} &= \begin{bmatrix} \bar{B} \cdots \bar{A}^{n-1}\bar{B} \end{bmatrix} \\
&= \begin{bmatrix} C^T \cdots (A^T)^{n-1}C^p \end{bmatrix} \\
&= \begin{bmatrix} B \\ \vdots \\ CA^{n-1} \end{bmatrix} \\
&= \mathcal{O}^T.
\end{aligned}
$$

Thus, the controllability matrix for $(\bar{A}, \bar{B})$ is the transpose of the observability matrix for (C, A). As the rank of a matrix and its transpose are the same, we have that $(\bar{A}, \bar{B})$ is controllable if and only iff (C, A) is observable. Thus, we can state the following result:

Theorem 3. *There exists a matrix L such that the eigenvalues of $A - LC$ can be placed arbitrarily if and only if the pair (C, A) is observable.*

The procedure for obtaining an L to place the eigenvalues of $A - LC$ is precisely the same as that considered in the state-feedback problem. For a system described by a transfer function

$$
G(s) = \frac{b_{n-1}s^{n-1} + \cdots b_0}{s^n + a_{n-1}s^{n-1} + \cdots a_0}
$$

the system can be written in *observable canonical form*, in which

$$
A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}, \qquad B = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix},
$$

and

$$
C = \begin{bmatrix} 0 \cdots 0 \; 1 \end{bmatrix}.
$$

As we did earlier, it is easy to check that the state-space representation is always observable. Moreover, if

$$
L = \begin{bmatrix} l_0 \\ l_1 \\ \vdots \\ l_{n-1} \end{bmatrix},
$$

then

$$A - LC = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 - l_0 \\ 1 & 0 & \cdots & 0 & -a_1 - l_1 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} - l_{n-1} \end{bmatrix}.$$

Moreover,

$$\det(sI - [A - LC]) = s^n + [a_{n-1} + l_{n-1}]s^{n-1} + \cdots + [a_0 + l_0].$$

Thus, the eigenvalues can be placed at the roots of

$$\Delta_d(s) = s^n + d_{n-1}s^{n-1} + \cdots + d_0,$$

with the choice

$$l_i = d_i - a_i, \qquad i = 0, \ldots, n-1.$$

If (C, A) is observable, but not in observable canonical form, then the procedure for obtaining an L is exactly like Ackermann's formula.

6.2 Building observer-based controllers

The previous section showed you how to obtain an observer that can make the estimation error approach zero asymptotically. However, our ultimate goal is to control the system. A natural *ad hoc* approach is to apply state-feedback control, but use the estimate of the state rather than the actual state. That is, instead of using the feedback control

$$u(t) = -Kx(t)$$

we use

$$u(t) = -K\tilde{x}(t),$$

where $\tilde{x}$ is the estimated state. Clearly, what we would like is for

$$e(t) = x(t) - \tilde{x}(t) \approx 0.$$

We will now show that this actually works. That is, we can make the system's states all go to zero. Note that the complete system is now defined by the states from the plant:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

and the observer

$$\dot{\tilde{x}}(t) = A\tilde{x} + Bu(t) + L\big(y(t) - C\tilde{x}(t) - Du(t)\big),$$

where, in both equations:

$$u(t) = -K\tilde{x}(t).$$

We can combine the system into one equation:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\tilde{x}}(t) \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A - BK - LC \end{bmatrix} \begin{bmatrix} x(t) \\ \tilde{x}(t) \end{bmatrix}$$

where we used the fact that $y(t) = Cx(t) + Du(t)$.

Determining where the eigenvalues of this 2×2 block-matrix as written is difficult. However, we can change the state variable. Rather than working with

$$\begin{bmatrix} x(t) \\ \tilde{x}(t) \end{bmatrix},$$

we work with

$$\begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ x(t) - \tilde{x}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}}_{T} \begin{bmatrix} x(t) \\ \tilde{x}(t) \end{bmatrix}.$$

In these coordinates, we have:

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} &= T \begin{bmatrix} \dot{x}(t) \\ \dot{\tilde{x}}(t) \end{bmatrix} \\ &= T \begin{bmatrix} A & -BK \\ LC & A - BK - LC \end{bmatrix} \begin{bmatrix} x(t) \\ x(t) \end{bmatrix} \\ &= T \begin{bmatrix} A & -BK \\ LC & A - BK - LC \end{bmatrix} T^{-1} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \\ &= \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}. \end{aligned}$$

Note that the eigenvalues of

$$\begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix},$$

are the union of the eigenvalues of $A - BK$ and $A - LC$. These have negative real parts if the controller gain K and the observer gain L are both chosen to make $A - BK$ and $A - LC$ asymptotically stable.

The controller is thus:

$$\dot{\tilde{x}}(t) = [A - BK - LC]\tilde{x}(t) + Ly(t) \tag{6.1}$$
$$u(t) = -K\tilde{x}(t) \tag{6.2}$$

which has transfer function:

$$C(s) = -K\big(sI - [A - BK - LC]\big)^{-1} L.$$

Note that this is an nth order strictly-proper controller.

6.3 Reduced order observers

In the previous section we demonstrated how to obtain observers to estimate the state $x(t)$. Clearly, this is useful if we do not know the states $x(t)$. However, in practice *some* states will be known. There is no point in estimating the states that are available. In this case, it is possible to obtain reduced order observers.

We will assume that the states are divided into two classes: available, $x_a(t)$, and unavailable, $x_u(t)$. We will assume that the states have been ordered such that

$$x(t) = \begin{bmatrix} x_a(t) \\ x_u(t) \end{bmatrix}.$$

Based on this division, we can write the state-space equation as

$$\begin{bmatrix} \dot{x}_a(t) \\ \dot{x}_u(t) \end{bmatrix} = \begin{bmatrix} A_{aa} & A_{au} \\ A_{ua} & A_{uu} \end{bmatrix} \begin{bmatrix} x_a(t) \\ x_u(t) \end{bmatrix} + \begin{bmatrix} B_a \\ B_u \end{bmatrix} u(t) \tag{6.3}$$

and the available states are

$$y(t) = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} x_a(t) \\ x_u(t) \end{bmatrix} = x_a(t).$$

The states that need to be estimated are those that are unavailable. Thus, we define the estimated unavailable states as $\tilde{x}_u(t)$ and form a system

$$\dot{\tilde{x}}_u(t) = A_{uu}\tilde{x}_u(t) + A_{ua}x_a(t) + B_u u(t) + \text{error driver}. \tag{6.4}$$

"Error driver" refers to the signal, derived from the output, which drives the state estimate. In the full-order observer, we had:

$$\text{error driver} = L\big(y(t) - C\tilde{x}(t) - Du(t)\big).$$

Note that the signal inside the brackets was selected to match the available signal, $y(t)$, and an estimate of this available signal $C\tilde{x}(t) + Du(t)$. We take the same approach for the reduced-order observer. We turn to the first set of rows in Eq. 6.3.

$$\underbrace{\dot{x}_a(t) - A_{aa}x_a(t) - B_a u(t)}_{\text{available}} = \underbrace{A_{au}x_u(t)}_{\text{unavailable}}.$$

We define:

$$\text{error driver} = L\Big(\underbrace{\dot{x}_a(t) - A_{aa}x_a(t) - B_a u(t)}_{\text{available}} - \underbrace{A_{au}\tilde{x}_u(t)}_{\substack{\text{estimated}\\ \text{unavailable}\\ \text{signal}}}\Big).$$

Note that the signal in the brackets is zero if the estimate is correct. We now incorporate this into Eq. 6.4:

$$\dot{\tilde{x}}_u = \big(A_{uu} - LA_{au}\big)\tilde{x}_u + \big(A_{ua} - LA_{aa}\big)x_a + L\dot{x}_a + \big(B_u - LB_a\big)u.$$

To make the observer estimate approach the real state we, once again, work with the error:

$$e_u(t) = x_u(t) - \tilde{x}_u(t)$$

and obtain:

$$\dot{e}_u(t) = \big(A_{uu} - LA_{au}\big)e_u(t),$$

which decays to zero asymptotically if the matrix

$$A_{uu} - LA_{au}$$

has eigenvalues with negative real parts. The eigenvalues of this matrix can be set arbitrarily if the pair (A_{au}, A_{uu}) is observable.

Finally, we ask what happens if we apply:

$$u(t) = -\begin{bmatrix} K_a & K_u \end{bmatrix} \begin{bmatrix} x_a(t) \\ x_u(t) \end{bmatrix}$$

to the system. Working with the state

$$\begin{bmatrix} x(t) \\ e_u(t) \end{bmatrix}$$

we have:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}_u(t) \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A_{uu} - LA_{au} \end{bmatrix} \begin{bmatrix} x(t) \\ e_u(t) \end{bmatrix},$$

which is asymptotically stable if the eigenvalues of the matrices $A - BK$ and $A_{uu} - LA_{au}$ all have negative real parts.

Using the fact that $y = x_a$, the control system is given by:

$$\begin{aligned} \dot{\tilde{x}}_u &= \Big(A_{uu} - LA_{au} - \big(B_u - LB_a\big)K_u\Big)\tilde{x}_u \\ &\quad + \Big(A_{ua} - LA_{aa} - \big(B_u - LB_a\big)K_a\Big)y + L\dot{y}, \\ u &= -K_a y - K_u \tilde{x}_u. \end{aligned}$$

The transfer function is

$$C(s) = -K_a - K_u(sI - \tilde{A})^{-1}[sL + \tilde{B}],$$

where

$$\tilde{A} = A_{uu} - LA_{au} - \big(B_u - LB_a\big)K_u,$$

and

$$\tilde{B} = A_{ua} - LA_{aa} - \big(B_u - LB_a\big)K_a.$$

Note that this is an $n_u = \dim(x_u)$th order proper (but not strictly proper) controller.