Earlier, we showed how to select the transfer function for a controller that can place the closed-loop poles given the plant's open-loop transfer function. One difficulty of working with polynomials, however, is that the numerical accuracy can quickly deteriorate as the order of the polynomial increases. State-space description of systems can lead to numerically superior algorithms. We know provide details on how state-feedback controller can be used to place closed-loop eigenvalues.

## 5.1 State-feedback controllers

Suppose that we have a system described by the state-space representation:

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(t) = x_0$$
$$y(t) = Cx(t),$$

where the state-vector  $x(t) \in \mathbf{R}^n$  and the input and output signals, u(t) and y(t) respectively, are scalars. This means that the matrices satisfy the following dimensionsal requirements:

$$A \in \mathbf{R}^{n \times n}, \quad B \in \mathbf{R}^{n \times 1}, \quad \text{and} \quad C \in \mathbf{R}^{1 \times n}.$$

For a given input, the solution is given by:

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\sigma)}Bu(\sigma) \,\mathrm{d}\sigma.$$

In particular, if the input  $u(t) \equiv 0$ , the state

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$$x(t) = e^{At} x_0 \tag{5.1}$$

will decay to zero asymptotically for any  $x_0$  if the eigenvalues of the matrix A all have negative real parts.

The response described by Eq. 5.1 represents an open-loop response. We can also consider a closed-loop response under the assumption that the input is given by state-feedback:

$$u(t) = -Kx(t).$$

Here  $K \in \mathbf{R}^{1 \times n}$  is a constant matrix. Under this control input the system is now described by

$$\begin{split} \dot{x}(t) &= [A - BK] x(t), \qquad x(t) = x_0 \\ y(t) &= C x(t), \end{split}$$

and the state response is

$$x(t) = e^{[A - BK]t} x_0. (5.2)$$

As above, the state will decay to zero asymptotically if the eigenvalues of A - BK all have negative real parts. This, of course, will be determined in part by the choice of K. The *state-feedback eigenvalue placement problem* is: given A and B and a monic *n*th order polynomial  $\Delta_d(s)$ , find a K that makes

$$\det(sI - [A - BK]) = \Delta_d(s)$$

In the next section we solve this problem.

### 5.2 Eigenvalue placement

We begin with a particular form of (A, B).

**Lemma 1.** Suppose that (A, B) is in controllable canonical form, and let

 $\Delta(s) = s^n + d_{n-1}s^{n-1} + \dots + d_0$ 

for arbitrary  $d_i$ , i = 0, ..., n - 1. Then there exists a

$$K = \begin{bmatrix} k_0 \cdots k_{n-1} \end{bmatrix}$$

such that

$$\det(sI - [A - BK]) = s^{n} + d_{n-1}s^{n-1} + \dots + d_{0}$$

*Proof.* Write out the matrix A - BK and note that it has the same format as A with the  $a_i$  replaced with  $a_i + k_i$ . Because

$$\det(sI - A) = s^{n} + a_{n-1}s^{n-1} + \dots + a_{0},$$

it follows that

$$\det(sI - A + BK) = s^{n} + (a_{n-1} + k_{n-1})s^{n-1} + \dots + (a_{0} + k_{0})$$

Thus, if we pick the  $k_i$  according to

 $k_i = d_i - a_i, \qquad i = 0, \dots, n - 1,$ 

we get the desired result.  $\Box$ 

**Theorem 2.** Suppose that (A, B) is controllable. There exists a K such that

$$\det(sI - [A - BK]) = s^{n} + d_{n-1}s^{n-1} + \dots + d_{0}$$

for arbitrary  $d_i$ , i = 0, ..., n - 1.

*Proof.* From Lemma 4 , if (A, B) is in controllable canonical form:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & 1 \\ -a_0 - a_1 - a_2 \cdots - a_{n-1} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

the matrix

$$K = \begin{bmatrix} k_0 \cdots k_{n-1} \end{bmatrix}$$

does the trick, if  $k_i = d_i - a_i$ , for  $i = 0, \ldots, n - 1$ .

Now, suppose that the pair is not in controllable canonical form. We will show that there exists an invertible matrix T such that

$$TAT^{-1} = \bar{A}, \qquad TB = \bar{B}$$

where  $\bar{A}$  and  $\bar{B}$  are in CCF.

Define

$$\mathcal{C} = \begin{bmatrix} A^{n-1}B \ A^{n-2}B \cdots AB \ B \end{bmatrix} \quad \text{and} \quad \bar{\mathcal{C}} = \begin{bmatrix} \bar{A}^{n-1}\bar{B} \ \bar{A}^{n-2}\bar{B} \cdots \bar{A}\bar{B} \ \bar{B} \end{bmatrix}.$$

Because both (A, B) and  $(\overline{A}, \overline{B})$  are controllable, the matrices C and  $\overline{C}$  are invertible (changing the order of the columns does not change the rank). Moreover, as in Lemma 4, the second of these matrices has the form

$$\bar{\mathcal{C}} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \star & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \star & \star & \star & \star & 1 & 0 \\ \star & \star & \star & \star & \star & 1 \end{bmatrix}$$

where  $\star$  corresponds to elements whose exact values are not needed.

The claim is that the matrix  $T = \overline{CC}^{-1}$  puts the matrix in the correct format. To show this, we need to show that

$$TB = \bar{B} \iff \bar{\mathcal{C}}\mathcal{C}^{-1}B = \bar{B} \tag{5.3}$$

and

$$TAT^{-1} = \bar{A} \iff \mathcal{C}^{-1}A\mathcal{C} = \bar{\mathcal{C}}^{-1}\bar{A}\bar{\mathcal{C}}^{-1}$$
(5.4)

For Eq. 5.3, note that

$$\mathcal{C}^{-1}B = \begin{bmatrix} 0\\0\\\vdots\\1 \end{bmatrix},$$

because  $C^{-1}C = I$  and B is the last column of C; thus, the product  $C^{-1}B$  gives the last column of the identity.

Now,

$$\bar{\mathcal{C}}\mathcal{C}^{-1}B = \bar{\mathcal{C}}\begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix} = \begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix} = \bar{B},$$

by the form of  $\bar{C}$ .

We now show Eq. 5.4. First of all, note that

$$A\mathcal{C} = \begin{bmatrix} A^n B \ A^{n-1} B \ \cdots \ A^2 B \ A B \end{bmatrix}$$
$$= \begin{bmatrix} -a_0 B - a_1 A B - \cdots - a_{n-1} A^{n-1} B \ A^{n-1} B \ A^{n-2} B \ \cdots \ A B \end{bmatrix}$$

where the last line came from application of the Cayley-Hamilton theorem.<sup>1</sup>

Now, note that the matrix AB is the *second last* column of C; that  $A^2B$  is the *third last* column of C, etc. Thus,

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$$\mathcal{C}^{-1} \left[ A^{n-1}B \ A^{n-2}B \ \cdots \ AB \right] = \begin{bmatrix} 0 \ 0 \ \cdots \ 0 \\ 1 \ 0 \ \vdots \ 0 \\ 0 \ 1 \ \vdots \ 0 \\ \vdots \ \vdots \ \vdots \ \vdots \\ 0 \ 0 \ \cdots \ 1 \end{bmatrix}.$$

Also,  $-a_0B - a_1AB - \dots - a_{n-1}A^{n-1}B$  is

$$-a_0 \times [n \text{th col. of } \mathcal{C}] - a_1 \times [(n-1) \text{st col. of } \mathcal{C}] - \cdots - a_{n-1} \times [1 \text{st of } \mathcal{C}]$$

and, therefore,

$$\mathcal{C}^{-1} \left( -a_0 B - a_1 A B - \dots - a_{n-1} A^{n-1} B \right) = \begin{bmatrix} -a_{n-1} \\ -a_{n-2} \\ \vdots \\ -a_0 \end{bmatrix}.$$

<sup>&</sup>lt;sup>1</sup> That is, if det $(sI - A) = s^n + a_{n-1}s^{n-1} + \dots + a_0$ , then  $A^n + a_{n-1}S^{n-1} + \dots + a_0I = 0$ , or  $A^n = -a_{n-1}S^{n-1} - \dots - a_0I$ .

Thus, the left-hand-side of Eq. 5.4 is

$$\mathcal{C}^{-1}A\mathcal{C} = \begin{bmatrix} -a_{n-1} \ 0 \ 0 \ \cdots \ 0 \\ -a_{n-2} \ 1 \ 0 \ \vdots \ 0 \\ \vdots \ 0 \ 1 \ \vdots \ 0 \\ -a_1 \ \vdots \ \vdots \ \vdots \ \vdots \\ -a_0 \ 0 \ 0 \ \cdots \ 1 \end{bmatrix}.$$

However, since the characteristic polynomial of A and  $\bar{A}$  are the same, following the exact same procedure leads to

$$\bar{\mathcal{C}}^{-1}\bar{A}\bar{\mathcal{C}} = \begin{bmatrix} -a_{n-1} \ 0 \ 0 \ \cdots \ 0 \\ -a_{n-2} \ 1 \ 0 \ \vdots \ 0 \\ \vdots \ 0 \ 1 \ \vdots \ 0 \\ -a_1 \ \vdots \ \vdots \ \vdots \ \vdots \\ -a_0 \ 0 \ 0 \ \cdots \ 1 \end{bmatrix}.$$

which confirms Eq. 5.4.

Note that, if  $\bar{K}$  is the state feedback matrix that makes

$$\det(sI - \bar{A} + \bar{B}\bar{K}) = s^n + d_{n-1}s^{n-1} + \dots + d_0,$$

then  $K = \overline{K}T$  makes

$$det(sI - A + BK) = det(sT^{-1}T - [T^{-1}\bar{A}T] + [T^{-1}\bar{B}][\bar{K}T])$$
  
=  $det(T^{-1}[sI - \bar{A} + \bar{B}\bar{K}]T)$   
=  $detT^{-1}det(sI - \bar{A} + \bar{B}\bar{K}) detT$   
=  $det(sI - \bar{A} + \bar{B}\bar{K})$   
=  $s^n + d_{n-1}s^{n-1} + \dots + d_0$ ,

as required.

*Remark 2.* This formula for the state feedback matrix is known as "Ackermann's formula." The Matlab commands acker and place find the required K for a given (A, B) and a given set of required closed-loop eigenvalues.

## 5.3 Tracking in state-space systems

Tracking external references in the state-space configuation is not much different from the case of polynomial systems. Once again, the idea is to augment the plant with a model of the external signal.

In what follows, we are considering systems of the form

$$\dot{x} = Ax + Bu \tag{5.5}$$

$$y = Cx \tag{5.6}$$

where we assume that  $A \in \mathbf{R}^{n \times n}$ . We also consider an external reference r. Our goal is for the tracking error

$$e(t) = r(t) - y(t)$$

to go to zero asymptotically for a predetermined r. We will first deal with steps and then look at more general inputs.

#### 5.3.1 Tracking steps

In this case, the external signal

$$r(t) = 1, \qquad t \ge 0.$$

We want this to go to zero asymptotically. One way of ensuring that is to define a new state:

$$x_{n+1}(t) = \int_0^t e(t) \,\mathrm{d}t;$$

equivalently,

$$\dot{x}_{n+1} = e(t) = r(t) - y(t).$$

In particular, if the system reaches steady-state, the derivatives of all states (including  $x_{n+1}$ ) go to zero. The only way for this to happen is for  $e(t) \rightarrow 0$ .

It follows that we need to ensure that all the states reach equilibrium. We do this by defining an aumented state

$$z(t) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} x \\ x_{n+1} \end{bmatrix}$$

Writing the state space description:

$$\dot{z} = \begin{bmatrix} \dot{x} \\ \dot{x}_{n+1} \end{bmatrix}$$
$$= \begin{bmatrix} Ax + Bu \\ r - Cx \end{bmatrix}$$
$$= \underbrace{\begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}}_{\bar{A}} \underbrace{\begin{bmatrix} x \\ x_{n+1} \end{bmatrix}}_{z} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\bar{B}} u + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\hat{B}} r.$$

Thus, the augmented system has the description

$$\dot{z} = \bar{A}z + \bar{B}u + \hat{B}r.$$

Notice that there are two inputs: the control input u, and the external input r. To stabilize the system, we seek a

 $\bar{K} = \begin{bmatrix} K \ k_I \end{bmatrix}$ 

such that the state-feedback input

 $u=-\bar{K}z$ 

can make the eigenvalues of the closed-loop system matrix

$$\bar{A}_K = \bar{A} - \bar{B}\bar{K}$$

stable. In fact, if  $(\bar{A}, \bar{B})$  is controllable, then the eigenvalues of  $\hat{A}_K$  can be set arbitrarily. Note, however, that just because (A, B) is controllable does not imply that  $(\bar{A}, \bar{B})$  is controllable.

Example 10. Consider the transfer function

$$G(s) = \frac{s}{(s+1)^2}.$$

It has a controllable canonical state space representation

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Clearly, this is (A, B) is controllable since the representation is in CCF. However, the augmented system has:

$$\bar{A} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & -1 & 0 \end{bmatrix},$$

and

$$\bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

The controllability matrix for this system is

$$\bar{\mathcal{C}} = \begin{bmatrix} \bar{B} \\ \bar{A}\bar{B} \\ \bar{A}^2\bar{B} \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix},$$

which is of rank 2 (the third column equals minus the first minus two times the second) and hence the augmented system is not controllable.  $\Box$ 

It is relatively easy to see why augmenting the system with an integrator renders the state-space system uncontrollable. The open-loop plant has a zero at s = 0. Adding an integrator, that is, a pole at zero induces a pole-zero cancellation. Hence, the augmented plant's numerator and denominator are no longer coprime. By the results of the previous chapter, we can expect that the closed-loop poles can not be set arbitrarily.

Using the state-feedback controller, the system will approach a steady-state, given by

$$\dot{z} = 0 = (\bar{A} - \bar{B}\bar{K})z + \hat{B}r \Rightarrow z = -(\bar{A} - \bar{B}\bar{K})^{-1}\hat{B}r.$$

We have taken the inverse of  $\overline{A} - \overline{B}\overline{K}$ . We know that this inverse exists, because the matrix is stable. This means that the eigenvalues all have negative real part eigenvalues, and hence can not be at zero.

Suppose that

$$(\bar{A} - \bar{B}\bar{K})^{-1} = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}.$$

It follows that

$$\begin{split} I &= (\bar{A} - \bar{B}\bar{K}) \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \\ &= \begin{bmatrix} A - BK_0 & -Bk_1 \\ -C & 0 \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \end{split}$$

In particular, from the (2,2) element of this product:

$$-C\Phi_{12} = 1.$$

Hence:

$$y = \begin{bmatrix} C & 0 \end{bmatrix} z$$
  
=  $-\begin{bmatrix} C & 0 \end{bmatrix} (\bar{A} - \bar{B}\bar{K})^{-1}\hat{B}r$   
=  $-\begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$   
=  $-C\Phi_{12}r$   
=  $r.$  (5.7)

Thus, the system is tracking the constant reference r.

#### 5.3.2 Tracking sinusoids

If you are not paying close attention, you might think that the argument presented above works for any reference input. After all, Eq. 5.7 says that the output y equals

the reference r, and we did not seem to use anywhere the fact that the reference was a step.

Unfortunately, if you did think this, you would be wrong. The problem is that, to arrive at Eq. 5.7, we needed to assume that the system reached an equilibrium (i.e. the derivatives of the states were zero). But this will not be the case in general.

The procedure for getting there is similar when the references are signals other than steps. However, the argument as to why steady-state tracking is achieved is somewhat more complicated. Notice that when the reference is a step, we define the state

$$\dot{x}_{n+1} = e(t).$$

In the Laplace transform domain, this is

$$X_{n+1}(s) = \frac{1}{s}E(s).$$

To track a sinusoid of frequency  $\omega_0$ , we once again augment the state of the system, but we do so with multiple states. In particular, we define states:

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} x_{n+1} \\ x_{n+2} \end{bmatrix} = \begin{bmatrix} x_{n+2} \\ -\omega_0^2 x_{n+1} + e \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} \begin{bmatrix} x_{n+1} \\ x_{n+2} \end{bmatrix} + \begin{bmatrix} 0 \\ r-y \end{bmatrix}$$

In the Laplace transform domain, this is equivalent to

$$X_{n+1}(s) = \frac{1}{s^2 + \omega_0^2} E(s).$$

Now the procedure is similar. We define a new state  $z \in \mathbf{R}^{n+2}$  according to:

$$z(t) = \begin{bmatrix} x\\ x_{n+1}\\ x_{n+2} \end{bmatrix}$$

and state-space description

$$\begin{split} \dot{z} &= \begin{bmatrix} \dot{x} \\ \dot{x}_{n+1} \\ \dot{x}_{n+2} \end{bmatrix} \\ &= \begin{bmatrix} Ax + Bu \\ x_{n+2} \\ -\omega_0^2 x_{n+1} + r - Cx \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} A & 0 & 0 \\ 0 & 0 & 1 \\ -C & -\omega_0^2 & 0 \end{bmatrix}}_{\bar{A}} \underbrace{\begin{bmatrix} x \\ x_{n+1} \\ x_{n+2} \end{bmatrix}}_{z} + \underbrace{\begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix}}_{\bar{B}} u + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\hat{B}} r. \end{split}$$

Once again, the augmented system has the description

 $\dot{z} = \bar{A}z + \bar{B}u + \hat{B}r.$ 

Now, if  $(\bar{A},\bar{B})$  is controllable, then there exists a

 $\bar{K} = \begin{bmatrix} K \ k_1 \ k_2 \end{bmatrix}$ 

such that

$$\bar{A}_K = \bar{A} - \bar{B}\bar{K}$$

is stable.

The error is

$$e = r - y = r - Cx$$

Thus, the whole state-space description of the system, using the external input r and the error e as the external output is

$$\dot{z} = \bar{A}_K z + \hat{B}r$$
$$e = \bar{C}z + r$$

with transfer function

$$\frac{E(s)}{R(s)} = \bar{C}(sI - \bar{A}_K)^{-1}\hat{B} + 1.$$

Once again, we can write

$$(sI - \bar{A}_K)^{-1} = \begin{bmatrix} \phi_{11} \ \phi_{12} \ \phi_{13} \\ \phi_{21} \ \phi_{22} \ \phi_{23} \\ \phi_{31} \ \phi_{32} \ \phi_{33} \end{bmatrix};$$

then

$$\frac{E(s)}{R(s)} = \bar{C}(sI - \bar{A}_K)^{-1}\hat{B} + 1$$
$$= \begin{bmatrix} -C \ 0 \ 0 \end{bmatrix} \begin{bmatrix} \varPhi_{11} \ \varPhi_{12} \ \varPhi_{13} \\ \varPhi_{21} \ \varPhi_{22} \ \varPhi_{23} \\ \varPhi_{31} \ \varPhi_{32} \ \varPhi_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 1$$
$$= 1 - C\varPhi_{13}.$$

From the (3,3) element of the product

$$(sI - \bar{A}_K)\Phi = I \quad \Rightarrow \quad \begin{bmatrix} sI - A & 0 & 0 \\ 0 & s & -1 \\ C & \omega_0^2 & s \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} \\ \Phi_{21} & \Phi_{22} & \Phi_{23} \\ \Phi_{31} & \Phi_{32} & \Phi_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we get:

$$C\Phi_{13} + \omega_0^2 \Phi_{23} + s\Phi_{33} = 1. \tag{5.8}$$

Moreover, from the (2,3) element of this product:

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 $s\Phi_{23} - \Phi_{33} = 0.$ 

Substituting this into Eq. 5.8 yields:

$$C\Phi_{13} = 1 - (\omega_0^2 + s^2)\Phi_{23}.$$

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Hence,

$$\frac{E(s)}{R(s)} = \bar{C}(sI - \bar{A}_K)^{-1}\hat{B} + 1$$
  
=  $-C\Phi_{13} + 1$   
=  $(\omega_0^2 + s^2)\Phi_{23} = \Gamma(s).$ 

Finally, we know that for a stable transfer function  $\Gamma(s)$ , a sinusoidal input leads to a steady-state sinusoidal output, with magnitude

$$|\Gamma(j\omega_0)| = |\Gamma(s)|_{s=j\omega_0} = (\omega_0^2 + s^2)\Phi_{23}|_{s=j\omega_0} = 0.$$

This shows that, in steady-state, the error goes to zero asymptotically.