

Pole placement using polynomial methods

Previously, we showed that a requirement of closed-loop systems is that the roots of the characteristic polynomial be inside a given region of the left-hand plane. Here we ask the question. Given a plant $G(s)$, which may include the plant $P(s)$ and feedback sensor $F(s)$, can we find a controller $C(s)$ that can place the roots of the characteristic polynomial in proscribed locations. This is known as the *pole-placement problem*.

4.1 Degree requirements

Suppose that we have a system described by the rational transfer function

$$G(s) = \frac{B(s)}{A(s)},$$

where

$$B(s) = b_n s^n + b_{n-1} s^{n-1} + \cdots + b_0,$$

and

$$A(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0.$$

We ask whether there exists a controller:

$$C(s) = \frac{P(s)}{L(s)},$$

where

$$P(s) = p_m s^m + p_{m-1} s^{m-1} + \cdots + p_0,$$

and

$$L(s) = l_m s^m + l_{m-1} s^{m-1} + \cdots + l_0,$$

such that the roots of the characteristic polynomial can be placed in arbitrary locations. Note that in this case, the “given” data is in terms of the coefficients of the plant. The degree of the controller, m , as well as the specific controller coefficients, $\{p_i\}_{i=0}^m$ and $\{l_i\}_{i=0}^m$, are to be chosen. Once we choose m , the characteristic polynomial is given by:

$$\Delta(s) = A(s)L(s) + B(s)P(s),$$

which is an $(n + m)$ th order polynomial. We can select $n + m$ desired closed-loop pole locations, leading to a desired characteristic polynomial:

$$\Delta_d(s) = d_{n+m}s^{n+m} + d_{n+m-1}s^{n+m-1} + \cdots + d_0,$$

and use our free parameters (the l_i and p_i) to satisfy

$$\Delta(s) = \Delta_d(s).$$

In the next section we will show when this problem has a solution and provide an algorithm for finding the required controller. Before doing so, we provide some examples.

Example 2. Consider the plant

$$G(s) = \frac{1}{s}.$$

Can we find a controller $C(s)$ that places all the closed-loop poles at -1 ?

Let us work with the simplest possible controller: $C(s) = k$. This is a controller of order $m = 0$. In this case, the characteristic polynomial is given by

$$\Delta(s) = s + k.$$

Clearly, by setting $k = -1$ we achieve

$$\Delta(s) = \Delta_d(s) = s + 1.$$

□

Example 3. We now repeat the problem above for the second order plant:

$$G(s) = \frac{1}{s^2}.$$

Once again, we first pick a static controller: $C(s) = k$. The characteristic polynomial will be order 2. Thus, we seek a gain k for which

$$\Delta(s) = s^2 + k = (s + 1)^2.$$

It is clear that there is no such k . However, we are not restricted to static controllers. Let us increase the order of the controller to $m = 1$:

$$C(s) = \frac{p_1 s + p_0}{l_1 s + l_0}.$$

With this controller, the degree of the characteristic polynomial is three, which means that the desired characteristic polynomial is:

$$\Delta_d(s) = (s + 1)^3 = s^3 + 3s^2 + 3s + 1.$$

Thus, we seek parameters p_1 , p_0 , l_1 , and l_0 such that

$$\Delta(s) = A(s)L(s) + B(s)P(s) = l_1 s^3 + l_0 s^2 + p_1 s + p_0 = \Delta_d(s).$$

Clearly, this is possible by setting

$$l_1 = 1, \quad l_0 = 3, \quad p_1 = 3, \quad \text{and} \quad p_0 = 1.$$

Thus, the controller

$$C(s) = \frac{3s + 1}{s + 3}$$

places all the closed-loop poles in the desired locations. \square

In the example above we saw that one needs a certain number of free parameters to be able to match all the coefficients of the characteristic polynomial. Note that if the plant has order n , and the controller has order m , then the characteristic polynomial is of order $n + m$ and there must be $n + m + 1$ coefficients in $\Delta_d(s)$. To set $\Delta(s) = \Delta_d(s)$ requires that each of these $n + m + 1$ coefficients be matched. Now, if the controller is of order m , there are $2(m + 1)$ “free” parameters, $m + 1$ from the numerator and $m + 1$ from the denominator. If we need the number of free parameters to be greater than or equal to the number of constraints, then

$$2(m + 1) \geq m + n + 1.$$

Thus, a requirement is that $m \geq n - 1$: the order of the plant must be at least the order of the plant minus one. Typically, we seek for the controller of least degree that achieves the requirement, and so we set $m = n - 1$.

Remark 1. Without loss of generality, the denominator polynomial, $A(s)$, and the characteristic polynomial can both be chosen to be monic. If this is the case, the controller denominator will also be monic.

4.2 Sylvester’s theorem

From above, we know that if we set $m \geq n - 1$, then there are enough degrees of freedom to solve the pole-placement problem. Is this enough? The answer is no, as the following example shows.

Example 4. Consider the plant $G(s) = \frac{s-2}{s^2-4}$. Find a controller that places all closed-loop poles at -1 .

Because the order of the plant is $n = 2$, from the argument above, we set $m = 1$:

$$C(s) = \frac{p_1 s + p_0}{l_1 s + l_0},$$

and the characteristic polynomial is of degree three. Thus

$$\Delta_d(s) = (s + 1)^3.$$

The characteristic polynomial is

$$\begin{aligned} \Delta(s) &= A(s)L(s) + B(s)P(s) \\ &= (s^2 - 4)(l_1 s + l_0) + (s - 2)(p_1 s + p_0) \\ &= l_1 s^3 + (l_0 + p_1)s^2 + (-4l_1 - 2p_1 + p_0)s - 4l_0 - 2p_0. \end{aligned}$$

To set this equal to $\Delta_d(s)$ we equate each of the four coefficients:

$$\begin{array}{ll} s^3 : & l_1 = 1 \\ s^2 : & l_0 + p_1 = 3 \\ s^1 : & -4l_1 - 2p_1 + p_0 = 3 \\ s^0 : & -4l_0 - 2p_0 = 0. \end{array}$$

We write these four equations in matrix form:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -4 & 0 & -2 & 1 \\ 0 & -4 & 0 & -2 \end{bmatrix}}_M \underbrace{\begin{bmatrix} l_1 \\ l_0 \\ p_1 \\ p_0 \end{bmatrix}}_{\theta} = \underbrace{\begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix}}_x.$$

This matrix equation $M\theta = x$ has a solution for arbitrary x if the matrix M is invertible; that is, if $\det M \neq 0$. Unfortunately, this is not the case here. You can easily check that the second column of M is a linear combination of the third and fourth columns (it equals the third plus two times the fourth). \square

We can look for a reason why the pole placement problem can not be solved for the plant of this example. Since

$$G(s) = \frac{s-2}{s^2-4} = \frac{s-2}{(s-2)(s+2)},$$

the characteristic polynomial will have:

$$\Delta(s) = (s-2)P(s) + (s-2)(s+2)L(s) = (s-2)(P(s) + (s+2)L(s)).$$

Thus, no matter what one chooses for the controller, the characteristic polynomial *must* have the root at 2. This is a consequence of the common root between the numerator and denominator of the plant. This is a general statement, known as *Sylvester's theorem*. In particular, consider the general plant $G(s)$ and controller $C(s)$. The corresponding matrix equation is $M\theta = x$ where

$$M = \begin{bmatrix} a_n & 0 & \cdots & 0 & b_n & 0 & \cdots & 0 \\ a_{n-1} & a_n & \cdots & 0 & b_{n-1} & b_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \ddots & a_n & b_1 & b_2 & \ddots & b_n \\ a_0 & a_1 & \ddots & a_{n-1} & b_0 & b_1 & \ddots & b_{n-1} \\ 0 & a_0 & \ddots & a_{n-2} & 0 & b_0 & \ddots & b_{n-2} \\ \vdots & 0 & \ddots & \vdots & \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & a_0 & 0 & \cdots & 0 & b_0 \end{bmatrix}, \quad \theta = \begin{bmatrix} l_{n-1} \\ l_{n-2} \\ \vdots \\ l_0 \\ p_{n-1} \\ p_{n-2} \\ \vdots \\ p_0 \end{bmatrix},$$

and

$$x = \begin{bmatrix} d_{2n-1} \\ \vdots \\ d_0 \end{bmatrix}.$$

The following theorem tells when the matrix M is invertible:

Theorem 1 (Sylvester's theorem). *The matrix M is invertible if and only if the polynomials $B(s)$ and $A(s)$ are coprime.*

Proof. Let us begin by assuming that $A(s)$ and $B(s)$ are not coprime. This means that there exists a common root which we designate at r :

$$A(s) = (s - r)A'(s), \quad \text{and} \quad B(s) = (s - r)B'(s),$$

where

$$A'(s) = a'_{n-1}s^{n-1} + a'_{n-2}s^{n-2} + \cdots + a'_0,$$

and

$$B'(s) = b'_{n-1}s^{n-1} + b'_{n-2}s^{n-2} + \cdots + b'_0.$$

By eliminating the common root, we get:

$$A(s)B'(s) - B(s)A'(s) = 0, \tag{4.1}$$

which, in matrix form is:

$$M\theta' = 0,$$

where

$$\theta' = \begin{bmatrix} b'_{n-1} \\ \vdots \\ b'_0 \\ -a'_{n-1} \\ \vdots \\ -a'_0 \end{bmatrix}$$

It follows that there exists a non-zero vector θ' that makes $M\theta' = 0$, which means that M not invertible.

The proof in the other direction follows similarly. Start with a vector θ' that makes $M\theta' = 0$ and create $A'(s)$ and $B'(s)$ such that Eq. 4.1 holds. It follows that

$$\frac{B(s)}{A(s)} = \frac{B'(s)}{A'(s)}.$$

The left-hand side of this equation involves two n th order polynomials; the right-hand side two $(n - 1)$ th order polynomials. It follows that the two polynomials in the left-hand must have a cancellation, which involves a common root.

Example 5. Consider the plant

$$G(s) = \frac{3}{(s+1)(s+3)} = \frac{3}{s^2 + 4s + 3}. \quad (4.2)$$

Find the simplest controller $C(s)$ such that the *dominant* closed-loop poles are at $-3 \pm j$. Any other poles should be at -12 .

Solution: The plant is of order $n = 2$. A proper controller of order 1 exists that can place the poles arbitrarily. It has the form:

$$C(s) = \frac{p_1 s + p_0}{l_1 s + l_0}$$

The characteristic polynomial has degree $n + m = 2 + 1 = 3$. Thus, we want to place the characteristic polynomial to be:

$$\begin{aligned} \Delta(s) &= [(s+3)^2 + 1](s+12) \\ &= (s^2 + 6s + 10)(s+12) \\ &= s^3 + 18s^2 + 82s + 120. \end{aligned}$$

To solve this we begin by forming the M matrix given by:

$$M = \begin{bmatrix} a_2 & 0 & b_2 & 0 \\ a_1 & a_2 & b_1 & b_2 \\ a_0 & a_1 & b_0 & b_1 \\ 0 & a_0 & 0 & b_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 3 & 4 & 3 & 0 \\ 0 & 3 & 0 & 3 \end{bmatrix}.$$

The closed-loop vector is:

$$x = \begin{bmatrix} d_{2n-1} \\ \vdots \\ d_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 18 \\ 82 \\ 120 \end{bmatrix}.$$

Now we solve for the controller:

$$M\theta = x \Rightarrow \theta = \begin{bmatrix} 1 \\ 14 \\ 23/3 \\ 26 \end{bmatrix}.$$

From this we obtain:

$$\begin{bmatrix} l_1 \\ l_0 \end{bmatrix} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 14 \end{bmatrix},$$

and

$$\begin{bmatrix} p_1 \\ p_0 \end{bmatrix} = \begin{bmatrix} \theta_3 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} 23/3 \\ 26 \end{bmatrix}.$$

Thus

$$C(s) = \frac{\frac{23}{3}s + 26}{s + 14}.$$

□

4.3 Strictly proper controllers

There may be cases when we seek to force the controller to have a specific format. One common requirement is that the controller be strictly proper. It is relatively easy to enforce this. In particular, suppose that we have an n th order plant $G(s)$ and that we seek an m th order strictly proper controller:

$$C(s) = \frac{P(s)}{L(s)} = \frac{p_{m-1}s^{m-1} + p_{m-2}s^{m-2} + \cdots + p_0}{l_ms^m + p_{m-1}s^{m-1} + \cdots + l_0}.$$

Notice how the requirement for strict properness uses up one of the free parameters (by setting $p_m = 0$). Thus, we now have $2m + 1$ free parameters, and still $n + m + 1$ constraints. The requirement that the number of free parameters exceed the constraints means that

$$2m + 1 \geq n + m + 1,$$

or $m \geq n$. It follows that a strictly proper controller will exist that places the closed-loop poles arbitrarily if $m \geq n$ and the polynomials $B(s)$ and $A(s)$ are coprime.

The procedure for finding the controller is virtually the same. We create matrices M , θ and x as above, except that there is one more column in M and one more row in θ — corresponding to the extra unknown l_n — and hence one more term in x corresponding to the increase in the degree of the characteristic polynomial. Thus,

$$M = \begin{bmatrix} a_n & 0 & \cdots & 0 & 0 & \cdots & 0 \\ a_{n-1} & a_n & \cdots & 0 & b_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \ddots & a_n & b_2 & \ddots & b_n \\ a_0 & a_1 & \ddots & a_{n-1} & b_1 & \ddots & b_{n-1} \\ 0 & a_0 & \ddots & a_{n-2} & b_0 & \ddots & b_{n-2} \\ \vdots & 0 & \ddots & \vdots & 0 & \ddots & \vdots \\ 0 & 0 & a_0 & 0 & b_0 \end{bmatrix}, \quad \theta = \begin{bmatrix} l_n \\ l_{n-1} \\ \vdots \\ l_0 \\ p_{n-1} \\ \vdots \\ p_0 \end{bmatrix}, \quad \text{and} \quad x = \begin{bmatrix} d_{2n} \\ \vdots \\ d_0 \end{bmatrix}.$$

Example 6. We repeat the previous example using a strictly proper controller placing the additional closed-loop poles at -15 .

As we need a strictly proper controller, the order of the controller is n . Notice now that there are five coefficients in the controller:

$$C(s) = \frac{p_1 s + p_0}{l_2 s^2 + l_1 s + l_0}.$$

We go through the steps as above:

$$M = \begin{bmatrix} a_2 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & 0 & b_2 & 0 \\ a_0 & a_1 & a_2 & b_1 & b_2 \\ 0 & a_0 & a_1 & b_0 & b_1 \\ 0 & 0 & a_0 & 0 & b_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 \\ 3 & 4 & 1 & 0 & 0 \\ 0 & 3 & 4 & 3 & 0 \\ 0 & 0 & 3 & 0 & 3 \end{bmatrix},$$

$$x = \begin{bmatrix} d_{2n} \\ \vdots \\ d_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 33 \\ 352 \\ 1350 \\ 1800 \end{bmatrix}.$$

Now, we solve for θ

$$\theta = \begin{bmatrix} l_2 \\ l_1 \\ l_0 \\ p_1 \\ p_0 \end{bmatrix} = M^{-1}x = \begin{bmatrix} 1 \\ 29 \\ 233 \\ 331/3 \\ 367 \end{bmatrix},$$

which results in the controller

$$C(s) = \frac{\frac{331}{3}s + 367}{s^2 + 29s + 233}.$$

□

4.4 Forcing controller structure into the pole placement algorithm

In the previous section we showed how certain requirements can be placed on the structure of the controller, specifically, requiring that the controller be strictly proper. In this section we show how other constraints can be made. As we saw earlier, the asymptotic rejection of persistent signals requires that the feedback loop include an internal model of the external signal. For steps, this means that the controller must have integral control. Similarly, for sinusoids of a given, known frequency, the loop must include imaginary axis poles at $\pm j\omega_0$. We will assume that these poles are not found in the plant model and thus must be supplied by the controller. We begin by showing how to include an integrator in the loop.

4.4.1 Integral action

If our controller must include an integrator, its transfer function must be able to be written as:

$$C(s) = \frac{1}{s} C'(s).$$

The transfer function $C'(s)$ includes all the unknown terms in the controller. Let us write this as

$$C'(s) = \frac{P'(s)}{L'(s)}$$

where

$$P'(s) = p'_m s^m + p'_{m-1} s^{m-1} + \cdots + p'_0,$$

and

$$L'(s) = l'_m s^m + l'_{m-1} s^{m-1} + \cdots + l'_0.$$

Note that the characteristic polynomial is

$$\begin{aligned} \Delta(s) &= A(s) [sL'(s)] + B(s)P'(s) \\ &= [sA(s)] L'(s) + B(s)P'(s). \end{aligned}$$

Writing $A'(s) = sA(s)$ and $B'(s) = B(s)$ we can set up an equivalent problem: Find the controller $C'(s)$ that places the closed-loop poles arbitrarily for the *augmented* plant

$$G'(s) = \frac{B'(s)}{A'(s)} = \frac{1}{s}G(s).$$

The procedure for incorporating the integrator into the controller is now straightforward. We include an integrator into the plant to form the augmented plant $G'(s)$. We solve for a controller $C'(s)$ for this augmented plant, and then include the integrator into the controller: $C(s) = \frac{1}{s}C'(s)$.

There are a couple of things to note about this technique. If the original plant is order n , then the augmented plant will be order $n + 1$. This means that the required order of the controller $C'(s)$ is one higher than had it been found directly on the plant. Moreover, because the implemented controller $C(s)$ includes an integrator, it is one extra degree higher. This suggests that the overall controller is two degrees higher. This is somewhat misleading, however. The reason is that the inclusion of the integrator into controller can make a proper $C(s)$ into a strictly proper $C'(s)$.

Example 7. Take the plant $G(s) = \frac{1}{(s-1)^2}$. A strictly proper controller that places the closed-loop poles automatically would be of order 2. If we want to include an integrator, we form the augmented plant $G'(s) = \frac{1}{s(s-1)^2}$ which is of order $n' = n + 1 = 3$. A proper, but not strictly proper, controller $C'(s)$ for this plant exists which is of order $m' = n' - 1 = 2$. The implemented controller is then of order $m' + 1 = 3$. \square

Example 8. We continue the earlier example in which we seek a controller for the plant $G(s)$ defined in Eq. 4.2. We now want a strictly proper controller to achieve zero steady-state tracking error for a unit step. The dominant closed-loop poles will be placed at $-3 \pm j$ and any other poles at -12 .

To solve this problem we augment the plant with the pole at zero. This leaves us with a third order plant:

$$G'(s) = \frac{3}{s(s+1)(s+3)} = \frac{3}{s^3 + 4s^2 + 3s} = \frac{b'_0}{a'_3s^3 + a'_2s^2 + a'_1s}.$$

There exists a second order controller that can place the poles. Even though this controller is proper and not strictly proper, the controller we implement will be strictly proper when we introduce the integrator. Thus, define:

$$C'(s) = \frac{p'_2s^2 + p'_1s + p'_0}{l'_2s^2 + l'_1s + l'_0}.$$

The desired closed-loop pole locations are the roots of the desired characteristic polynomial

$$\begin{aligned} \Delta_d(s) &= ((s+3)^2 + 1)(s+12)^3 \\ &= s^5 + 42s^4 + 658s^3 + 4,680s^2 + 14,688s + 17,280 \\ &= d_5s^5 + d_4s^4 + d_3s^3 + d_2s^2 + d_1s + d_0. \end{aligned}$$

The matrix equation we solve is

$$\begin{bmatrix} a'_3 & 0 & 0 & b'_3 & 0 & 0 \\ a'_2 & a'_3 & 0 & b'_2 & b'_3 & 0 \\ a'_1 & a'_2 & a'_3 & b'_1 & b'_2 & b'_3 \\ a'_0 & a'_1 & a'_2 & b'_0 & b'_1 & b'_2 \\ 0 & a'_0 & a'_1 & 0 & b'_0 & b'_1 \\ 0 & 0 & a'_0 & 0 & 0 & b'_0 \end{bmatrix} \begin{bmatrix} p'_2 \\ p'_1 \\ p'_0 \\ l'_2 \\ l'_1 \\ l'_0 \end{bmatrix} = \begin{bmatrix} d_5 \\ d_4 \\ d_3 \\ d_2 \\ d_1 \\ d_0 \end{bmatrix},$$

or

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 & 0 \\ 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & 3 & 4 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} p'_2 \\ p'_1 \\ p'_0 \\ l'_2 \\ l'_1 \\ l'_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 42 \\ 658 \\ 4,680 \\ 14,688 \\ 17,280 \end{bmatrix}$$

which leads to

$$C'(s) = \frac{\frac{2,554}{3}s^2 + 4,393s + 5,760}{s^2 + 38s + 503}.$$

Incorporating the integrator into the controller gives:

$$C(s) = \frac{\frac{2,554}{3}s^2 + 4,393s + 5,760}{s^3 + 38s^2 + 503s}.$$

□

4.4.2 Rejecting sinusoids

The procedure for rejecting sinusoids is nearly identical, except that we must include two augmented poles, at $\pm j\omega_0$. This means that the augmented plant:

$$G'(s) = \frac{1}{s^2 + \omega_0^2} G(s)$$

has degree $n' = n + 2$. The controller $C'(s)$ can then be obtained for this plant, and the controller

$$C(s) = \frac{1}{s^2 + \omega_0^2} C'(s)$$

is implemented. Note again that because the inclusion of the term $s^2 + \omega_0^2$ into the controller, the controller $C'(s)$ can be improper while still ensuring that $C(s)$ is proper or even strictly proper.

Example 9. As a final example, we desire to build a strictly proper controller for the plant in Eq. 4.2 that achieves zero steady-state tracking error for a sinusoid of frequency 1 rad/s and that places the dominant closed-loop poles at $-3 \pm j$ with any other closed-loop poles at -12 .

Because we want to reject sinusoids of frequency 1 rad/s, we should augment the plant with two poles at $\pm j$. Since we want a strictly-proper controller but there are two of these poles, the controller $C'(s)$ that we solve for need not be proper. The dimension of the augmented plant is $n' = n + 2 = 4$. If the controller has a degree n_1 numerator and a degree n_2 denominator then the controller will have $n_1 + n_2 + 2$ degrees of freedom and the closed-loop polynomial will have $(n + 2) + n_2 + 1$ coefficients. Thus, to ensure that the degrees of freedom is at least as high as the number of constraints we need: $n_1 \geq n + 1 = 3$. The denominator can have one less degree, so $n_2 = 2$. Thus, the controller can have the form

$$C'(s) = \frac{P'(s)}{L'(s)} = \frac{p_2 s^2 + p_1 s + p_0}{l_1 s + l_0}.$$

Let us now solve for these coefficients.

We will illustrate how this can be done using Matlab commands. We begin by defining the coefficients of the numerator and denominator polynomials:

```
>> A=[1 4 3];
>> B=[0 0 3];
```

We now augment the plant with the two poles. Polynomial multiplication can be done using the `conv` command (which stands for convolution):

```
>> Apr=conv(A,[1 0 1]);
>> Bpr=[0 B];
```

We are ready to form the matrix M :

```
>> M=[Apr 0 0; 0 Apr 0; 0 0 Apr; Bpr 0 0 0; ...
      0 Bpr 0 0; 0 0 Bpr 0; 0 0 0 Bpr]';
```

The vector x comes from the polynomial that has roots at the desired closed-loop pole locations. Once again, we use nested `conv` commands to obtain this:

```
>> x=conv([1 12],conv(conv([1 6 10],[1 12]),...
      conv([1 12],[1 12])));
```

The “prime” at the end of the last command is Matlab’s version of complex conjugate transpose. It is used here to make x a column vector.

We now solve for θ and obtain the controller $C'(s) = P'(s)/L'(s)$:

```
>> theta=M\x;
```

Note that the matrix M does not have to be inverted. In general, this gives more accurate answers.

```
>> Lprime=theta(1:3)'
Lprime =
     1     50    958
>> Pprime=theta(4:7)'
    1.0e+004 *
     0.2847     2.2271     6.3185     6.8162
```

Finally, we augment the controller with the two imaginary axis poles:

```
>> L=conv(Lprime,[1 0 1]);
>> P=[0 Pprime];
>> C=tf(P,L)

Transfer function:
2847 s^3 + 22271 s^2 + 6.318e004 s + 68162
-----
s^4 + 50 s^3 + 959 s^2 + 50 s + 958
```

You can check that the vector `Pprime` is actually:

$$P' = \frac{1}{3} [8, 540 \ 66, 813 \ 189, 554 \ 204, 486]$$

which means that the controller can be written as

$$C(s) = \frac{8,540s^3 + 66,813s^2 + 189,554s + 204,486}{3(s^4 + 50s^3 + 959s^2 + 50s + 958)}.$$

We perform a final check to ensure that the closed-loop poles are in the right place:

```
>> roots(conv(A,L)+conv(B,P))
ans =
-12.0017 + 0.0017i
-12.0017 - 0.0017i
-11.9983 + 0.0017i
-11.9983 - 0.0017i
-3.0000 + 1.0000i
-3.0000 - 1.0000i
```

reveals that the desired closed-loop poles have (almost) been obtained, though you should note that numerical noise creeps into the problem. \square