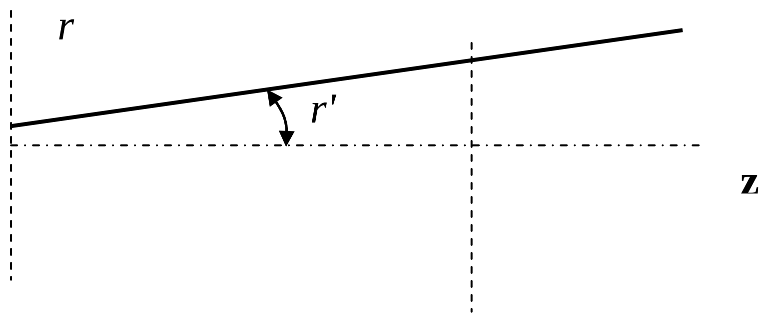
LECTURE 9
RAY TRACING:

In paraxial optics we consider the optical rays propagating very close to the axis. (no skewed rays) The ray is represented by the elevation r and angle r' at each point. Therefore we can use the column-vector to describe it as

$$\begin{vmatrix} r \\ r' \end{vmatrix}$$



Now, since all the equations are linear we can use the matrices to conenct them

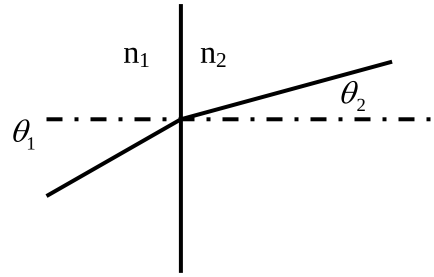
$$\begin{vmatrix} r_2 \\ r_2' \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \end{vmatrix} \begin{vmatrix} r_1 \\ r_1' \end{vmatrix}$$

Then if have a large system all we need to do is to multiply the matrices.

Important matrices.

Free Space -translation

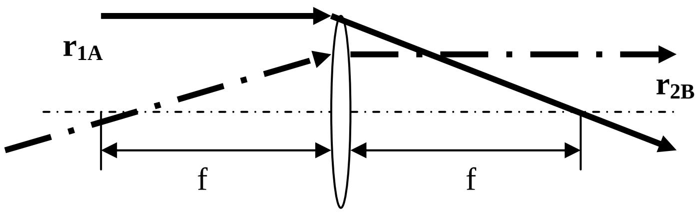
$r_2' = r_1'$; $r_2 = r_1 + r_1' d$ or $\begin{vmatrix} r_2 \\ r_2' \end{vmatrix} = \begin{vmatrix} 1 & d \\ 0 & 1 \end{vmatrix} \begin{vmatrix} r_1 \\ r_1' \end{vmatrix}$ thus translation matrix is $T_d = \begin{vmatrix} 1 & d \\ 0 & 1 \end{vmatrix}$



Refraction at the plane interface for paraxial angles –from Snell's law $n_1 \sin\theta_1 = n_2 \sin\theta_1$ for small angles we can write

$$r_2' = \frac{n_1}{n_2} r_1'; \quad r_2 = r_1 \quad \text{or} \quad \begin{vmatrix} r_2 \\ r_2' \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ n_1/n_2 & 1 \end{vmatrix} \begin{vmatrix} r_1 \\ r_1' \end{vmatrix} \quad \text{thus reflection matrix is } R = \begin{vmatrix} 1 & 0 \\ n_1/n_2 & 1 \end{vmatrix}$$

Thin Lens – the properties are that it collects all the rays parallel to the axis into the focus – in both directions, i.e.



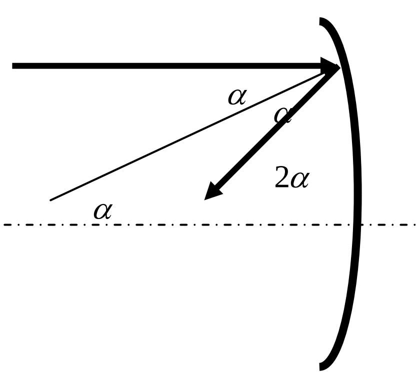
From the geometry:

for ray A: $$\begin{aligned} r_{2A} &= r_{1A} = Ar_{1A} + Br_{1A}' = Ar_{1A} \\ r_{2A}' &= -r_{1A}/f = Cr_{1A} + Dr_{1A}' = Cr_{1A} \end{aligned} \quad \text{and } A = 1;\ C = -1/f$$

for ray B: $$\begin{aligned} r_{2B} &= r_{1B} = Ar_{1B} + Br_{1B}' = r_{1B} + Br_{1B}'; \\ r_{2B}' &= 0 = Cr_{1B} + Dr_{1B}' = -\frac{1}{f} r_{1A} + D\frac{r_{1A}}{f} \end{aligned} \quad \text{thus } B = 0;\ D = 1$$

and we have the matrix $L_f = \begin{vmatrix} 1 & 0 \\ -1/f & 1 \end{vmatrix}$

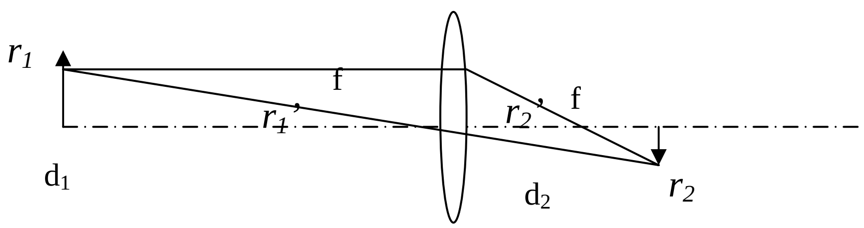
Finally, consider the *Mirror* with focal lens R/2



and we have $M_R = \begin{vmatrix} 1 & 0 \\ -2/R & 1 \end{vmatrix}$

Note that AD-BC=1 for all matrices.

Consider using the ray-tracing matrix for imaging with the thin lens



The matrix is

$$T_{d_1} L_f T_{d_2} = \begin{vmatrix} 1 & d_2 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{vmatrix} \begin{vmatrix} 1 & d_1 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & d_2 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & d_1 \\ -\frac{1}{f} & 1 - \frac{d_1}{f} \end{vmatrix} =$$

$$= \begin{vmatrix} 1 - \frac{d_2}{f} & d_1 + d_2 - \frac{d_1 d_2}{f} \\ -\frac{1}{f} & 1 - \frac{d_1}{f} \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \end{vmatrix}$$

but for the imaging it is necessary that all the rays converge, no matter what is the angle, i.e. the height of the image, r_2 should not depend on the angle r_1'. That means $B=0$, and the lens equation is

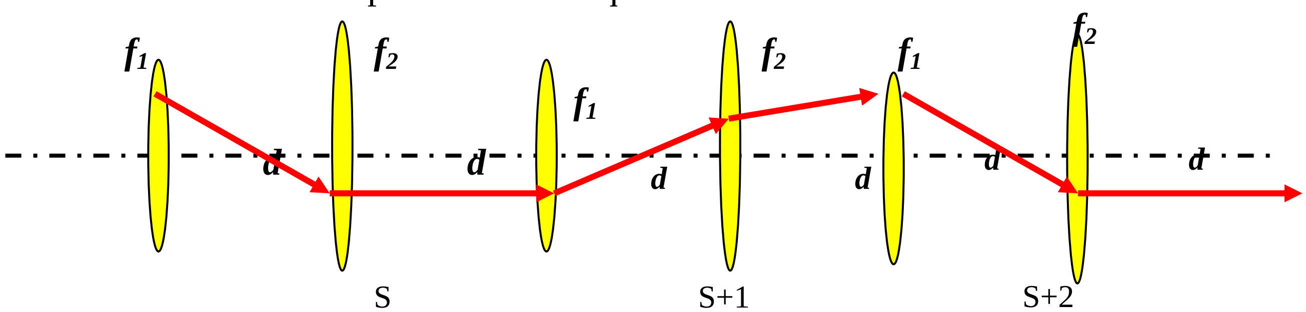
$$\frac{1}{d_1} + \frac{1}{d_2} = \frac{1}{f}$$

We can now obtain $r_2 = \left(1 - \frac{d_2}{f}\right) r_1 = M r_1$ - linear magnification and

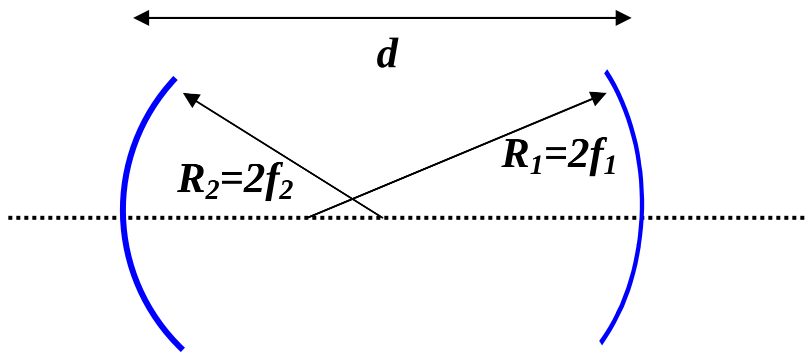
$r_2^{'} = \left(1 - \frac{d_1}{f}\right) r_1^{'} = M_\alpha r_1^{'}$ -angular magnification. Note that $\mathbf{M \times M_\alpha = 1}$.

Optical cavities

Consider the periodic lens sequence



This sequence is equivalent to a cavity



Write the matrix for the unit section:

$$T_d L_{f_1} T_d L_{f_2} = \begin{vmatrix} 1 & d \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{vmatrix} \begin{vmatrix} 1 & d \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{vmatrix} = \begin{vmatrix} 1 & d \\ -\frac{1}{f_1} & 1 - \frac{d}{f_1} \end{vmatrix} \begin{vmatrix} 1 & d \\ -\frac{1}{f_2} & 1 - \frac{d}{f_2} \end{vmatrix} =$$

$$\begin{vmatrix} 1 - \frac{d}{f_2} & d + d\left(1 - \frac{d}{f_2}\right) \\ -\frac{1}{f_1} - \frac{1}{f_1}\left(1 - \frac{d}{f_1}\right) & \left(1 - \frac{d}{f_1}\right)\left(1 - \frac{d}{f_2}\right) - \frac{d}{f_1} \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \end{vmatrix}$$

Let us make a second-order difference equation:

$$
\begin{aligned}
&r_{s+1} = Ar_s + Br_s^{'};\ r_s^{'} = B^{-1}(r_{s+1} - Ar_s) \\
&r_{s+1}^{'} = B^{-1}(r_{s+2} - Ar_{s+1}) = Cr_s + Dr_s^{'} = Cr_s + DB^{-1}(r_{s+1} - Ar_s) \text{ mult by B} \\
&r_{s+2} - Ar_{s+1} = BCr_s + Dr_{s+1} - ADr_s \text{ use } AD - BC = 1 \\
&r_{s+2} - (A+D)r_{s+1} + r_s = 0
\end{aligned}
$$

We look for the periodic bounded solution $r_s = r_0 \cos(sx + \varphi_0) = \mathrm{Re}[\tilde{r}_0 e^{jsx}]$ - substitute it to get

$$e^{j2x} - (A+D)e^{jx} + 1 = 0$$

$$e^{jx} = \frac{A+D}{2} \pm j\sqrt{1 - \left(\frac{A+D}{2}\right)^2} = \cos\theta \pm j\sin\theta$$

Two complex conjugate solutions – therefore

$$r_s = r_{\max}\cos(s\theta + \varphi_0)$$

Stability:

$$\cos\theta = \frac{A+D}{2};\ \text{for real } \theta \ \left|\cos\theta\right| < 1 \text{ or } -1 \le \frac{A+D}{2} \le 1 \text{ or } 0 \le \frac{A+D+2}{4} \le 1$$

Now

$$
\begin{aligned}
\frac{A+D+2}{4} &= \frac{1}{4}\left[\left(1 - \frac{d}{f_2}\right) + \left(1 - \frac{d}{f_1}\right)\left(1 - \frac{d}{f_2}\right) - \frac{d}{f_1} + 2\right] = \\
&= \frac{1}{4}\left[4 - \frac{2d}{f_1} - \frac{2d}{f_2} + \frac{d^2}{f_1 f_2}\right] = \left(1 - \frac{d}{2f_1}\right)\left(1 - \frac{d}{2f_2}\right) = \left(1 - \frac{d}{R_1}\right)\left(1 - \frac{d}{R_2}\right) = g_1 g_2
\end{aligned}
$$

Thus we have stability condition

$$0 \le g_1 g_2 < 1$$