Lecture 8

Fabry-Perot Cavity Modes

Now that we have studied the methods for creating population inversion and gain we shall look into the methods of obtaining a feedback. In RF this is done using 3D resonant cavities. Consider for example a 3D box of size $L_x \times L_y \times L_z$ with metal walls. Since the electric field at the boundaries must be equal to zero the solutions for the wave equation are

$$E_{mpq}(x,y,z,t) = E_0 \sin\left(m\frac{\pi}{L_x}x\right)\sin\left(p\frac{\pi}{L_y}y\right)\sin\left(q\frac{\pi}{L_z}z\right)e^{-j\omega t}$$

where

$$m^2\frac{\pi^2}{L_x^2} + p^2\frac{\pi^2}{L_y^2} + q^2\frac{\pi^2}{L_z^2} = \frac{\omega^2}{c^2}n^2$$

The number of modes in the frequency interval from ω to $\omega+\delta\omega$ is then

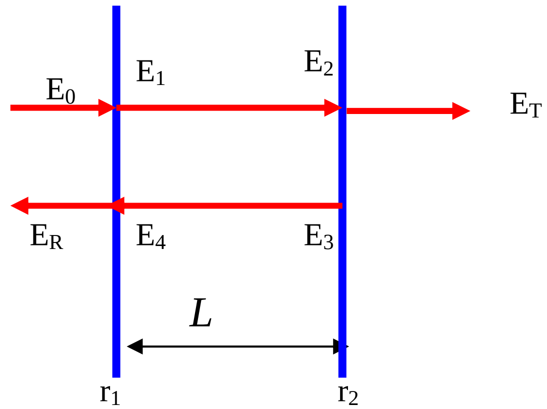
$$N_\omega = \rho(\omega)Vd\omega = \frac{n^3\omega^2 V}{\pi^2 c^3}d\omega = \frac{2\pi n^3 V}{\lambda^3}\frac{d\omega}{\omega};$$

The frequency interval between two adjacent modes is, therefore roughly

$$\Delta\omega \simeq \frac{\omega}{2\pi}\left(\frac{\lambda}{nL}\right)^3$$

In the RF one can have $L \sim \lambda$, but in optics, even for a relatively short cavities L is order of cm thus $\Delta\omega \sim 10^{-13}\omega$, i.e.$\Delta\nu \sim 10^{-13}\nu \sim 20$Hz. The modes are very close to each other; due to the broadening these modes are “smeared” into continuous spectrum and cavity has no frequency selectivity.

Therefore, one needs to use an “open” or one-dimensional resonator such as Fabry-Perot type:



Let us find the amplitude of the reflected field

$$E_1 = r_1 E_4 + t_1 E_0$$

$$E_4 = E_3 e^{j\theta} = r_2 E_2 e^{j\theta} = r_2 E_1 e^{j2\theta} = r_2 r_1 E_4 e^{j2\theta} + r_2 t_1 E_0 e^{j2\theta}$$

where $\theta = kL = 2\pi \nu n L / c = \pi \nu \tau_{rt}$;

$\tau_{rt} = 2nL/c -$ round trip time.

$$E_4 = \frac{r_2 t_1 E_0 e^{j2\theta}}{1 - r_2 r_1 e^{j2\theta}}$$

$$E_R = r_1' E_0 + t_1 E_4 = r_1' E_0 + \frac{r_2 t_1^2 E_0 e^{j2\theta}}{1 - r_2 r_1 e^{j2\theta}} = \frac{r_1' - r_2 r_1 r_1' e^{j2\theta} + r_2 t_1^2 e^{j2\theta}}{1 - r_2 r_1 e^{j2\theta}} E_0 = \frac{r_1' + r_2 e^{j2\theta}}{1 - r_2 r_1 e^{j2\theta}} E_0$$

since $r_1' = -r_1,\ r_1^2 + t_1^2 = 1$

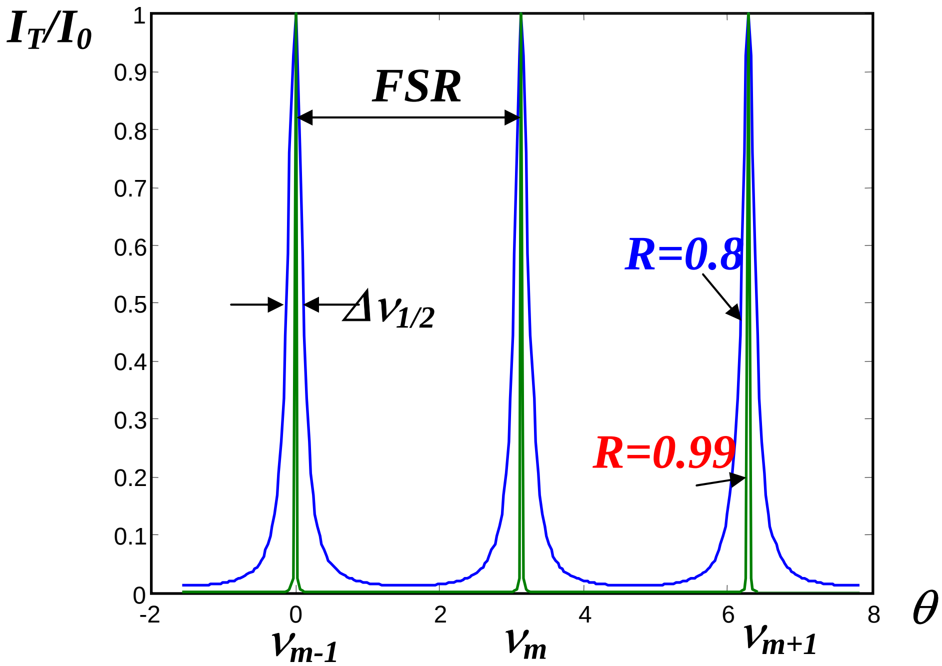
What about the intensity of the reflected field?

$$I_R = I_0 \frac{r_1' + r_2 e^{j2\theta}}{1 - r_2 r_1 e^{j2\theta}} \frac{r_1' + r_2 e^{-j2\theta}}{1 - r_2 r_1 e^{-j2\theta}} = \frac{r_1^2 + r_2^2 + 2 r_1' r_2 \cos 2\theta}{1 + r_1^2 r_2^2 - 2 r_1 r_2 \cos 2\theta} I_0 = \frac{R_1 + R_2 - 2\sqrt{R_1 R_2}\cos 2\theta}{1 + R_1 R_2 - 2\sqrt{R_1 R_2}\cos 2\theta} I_0$$

Transmission?

$$I_T = I_0 - I_R = \left[1 - \frac{R_1 + R_2 - 2\sqrt{R_1R_2}\cos 2\theta}{1 + R_1R_2 - 2\sqrt{R_1R_2}\cos 2\theta}\right] I_0 =$$

$$\frac{(1-R_1)(1-R_2)}{1 + R_1R_2 - 2\sqrt{R_1R_2}\cos 2\theta} I_0 = \frac{(1-R_1)(1-R_2)}{\left(1-\sqrt{R_1R_2}\right)^2 + 4\sqrt{R_1R_2}\sin^2\theta} I_0$$



Consider special cases:

$$\theta = m\pi \quad I_R = \frac{\left(\sqrt{R_1} - \sqrt{R_2}\right)^2}{\left(1-\sqrt{R_1R_2}\right)^2} I_0 \text{ if } R_1 = R_2 = R \ I_R = 0;$$

$$I_T = \frac{(1-R_1)(1-R_2)}{\left(1-\sqrt{R_1R_2}\right)^2} I_0 \text{ if } R_1 = R_2 = R \ I_T = 1$$

$$\theta = m\pi + \frac{\pi}{2} \;\; I_R = \frac{\left(\sqrt{R_1} + \sqrt{R_2}\right)^2}{\left(1 + \sqrt{R_1 R_2}\right)^2} I_0 \text{ if } R_1 = R_2 = R \;\; I_R = \frac{4R}{\left(1+R\right)^2}$$

$$I_T = \frac{(1-R_1)(1-R_2)}{\left(1+\sqrt{R_1 R_2}\right)^2} I_0 \text{ if } R_1 = R_2 = R \;\; I_T = \frac{\left(1-R\right)^2}{\left(1+R\right)^2}$$

The cavity is resonant at frequencies $\nu_m = m/\tau_{rt}$.

Let us find FWHM

$$I_T(\sin\theta_\pm) = \frac{1}{2} I_T(0)$$

$$\frac{(1-R_1)(1-R_2)}{\left(1-\sqrt{R_1 R_2}\right)^2 + 4\sqrt{R_1 R_2}\sin^2\theta_\pm} = \frac{1}{2}\frac{(1-R_1)(1-R_2)}{\left(1-\sqrt{R_1 R_2}\right)^2}$$

$$4\sqrt{R_1 R_2}\sin^2\theta_\pm = \left(1-\sqrt{R_1 R_2}\right)^2$$

$$\sin\theta_\pm = \frac{1-\sqrt{R_1 R_2}}{2\sqrt[4]{R_1 R_2}} = \pi\Delta\nu_{1/2} nL/c;$$

$$\Delta\nu_{1/2} = \frac{c}{2nL}\frac{1-\sqrt{R_1 R_2}}{\pi\sqrt[4]{R_1 R_2}}$$

Introduce the Free Spectral Range (FSR) as the difference between two resonant modes

$$FSR = \nu_{m+1} - \nu_m = \tau_{rt}^{-1} = \frac{c}{2nL}$$

Then $\Delta\nu_{1/2} = \frac{c}{2nL}\frac{1-\sqrt{R_1 R_2}}{\pi\sqrt[4]{R_1 R_2}} = FSR/F \qquad F = \frac{\pi\sqrt[4]{R_1 R_2}}{1-\sqrt{R_1 R_2}}$ – **finesse**

We can re-write the expression for the cavity transmission

$$I_T = \frac{(1-R_1)(1-R_2)}{\left(1-\sqrt{R_1R_2}\right)^2 + 4\sqrt{R_1R_2}\sin^2\left(\frac{\pi\nu}{FSR}\right)} I_0 = \frac{I_{\max}}{1+\left(\frac{2F}{\pi}\right)^2 \sin^2\left(\frac{\pi\nu}{FSR}\right)}$$

The last expression is correct even when additional losses are introduced into the cavity – one simply needs to re-define the Finesse.

We can also introduce Q of the cavity

$$Q = \frac{\nu}{\Delta\nu_{1/2}} = \frac{2\pi nL\sqrt[4]{R_1R_2}}{\left(1-\sqrt{R_1R_2}\right)\lambda_0} = F\,\frac{2nL}{\lambda_0}$$

PHOTON LIFETIME

Let us say the density of photons at one end of the cavity is N_p. After one round trip that takes time $\tau_{rt} = \frac{2nL}{c} = (FSR)^{-1}$ the density of photons becomes $N(t+\tau_{rt}) = N(t)R_1R_2$.

Use Taylor expansion

$$N_p(t+\tau_{rt}) = N_p(t) + \frac{dN_p}{dt}\tau_{rt} = N_p(t)R_1R_2$$

$$\frac{dN_p}{dt} = -(1-R_1R_2)\frac{N_p(t)}{\tau_{rt}} = -\frac{N_p(t)}{\tau_p}$$

$$\tau_p = \frac{\tau_{rt}}{1-R_1R_2} = \frac{2nL}{c(1-R_1R_2)} \approx \frac{2nL}{c\ln\frac{1}{R_1R_2}}$$

$$\Delta\omega_{1/2} = 2\pi\Delta\nu_{1/2} = \frac{2c}{2nL}\frac{1-\sqrt{R_1R_2}}{\sqrt[4]{R_1R_2}} \approx \frac{2c}{2nL}\left(1-\sqrt{1-(1-R_1R_2)}\right) \approx \frac{c(1-R_1R_2)}{2nL} = \tau_p^{-1}$$

$$\tau_p = \frac{\tau_{rt}}{2\pi}F$$

Assume there are additional losses per round trip, then

$$\tau_p = \frac{\tau_{rt}}{1 - R_1 R_2 (1 - Loss)} \approx \frac{2nL}{c(\ln \frac{1}{R_1 R_2} + 2\alpha L)};\ \alpha - \text{loss per unit length}$$

$$\tau_p^{-1} = \Delta\omega_{1/2} = \frac{c}{2nL} \ln \frac{1}{R_1 R_2} + \frac{\alpha c}{n}$$

The problem with Fabri-Perot resonator is that it does not confine the mode in the transverse direction…