Lecture 4

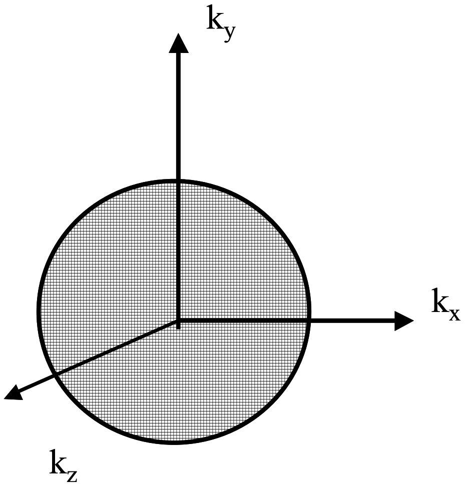
Photons

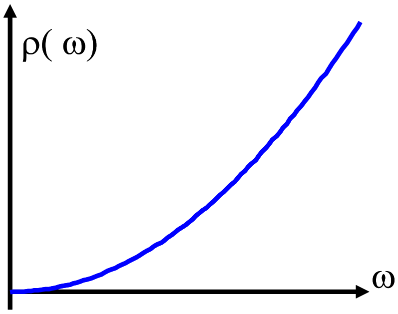
According to quantum mechanics the electro-magnetic radiation is quantized – the energy E_ω of the electro-magnetic field can be written as $E_\omega = n_\omega \hbar\omega$ where $\hbar = 1.02\times 10^{-34} J\cdot s$ and n_ω is a number of photons of frequency ω. How do we find the energy density of radiation at a given temperature T?
First of all, let us find density of electro-magnetic modes.
Consider a large metal cavity in the shape of cube with side *L*. From boundary conditions the field is

$$E(x,y,z,t) = E_0 \begin{matrix}\cos(k_x x)\\ \sin(k_y x)\end{matrix} \cdot \begin{matrix}\cos(k_y y)\\ \sin(k_y y)\end{matrix} \cdot \begin{matrix}\cos(k_z z)\\ \sin(k_z z)\end{matrix} e^{-j\omega t}$$

where $k_x = m_x \dfrac{\pi}{L},\ k_y = m_y \dfrac{\pi}{L},\ k_x = m_z \dfrac{\pi}{L}$

From wave equation $k^2 = k_x^2 + k_y^2 + k_z^2 = n^2 \dfrac{\omega^2}{c^2}$ -not every wave vector, and thus not every frequency is allowed.





Each "mode" occupies a "volume in k-space" $\Delta V_k = \dfrac{\pi^3}{L^3}$

The number of modes whose wave vector is less than some *k* is

$$\mathrm{N}(k) = 2\times\frac{4}{3}\pi k^3 \times \frac{1}{8}\Big/ \Delta V_k = \frac{1}{3\pi^2}k^3 L^3$$

The number of modes per unit volume whose frequency is less than $\omega = ck/n$

$$N_\omega = \frac{1}{3\pi^2} n^3 \frac{\omega^3}{c^3}.$$

The density of modes (per unit volume per frequency interval is

$$\rho(\omega) = dN_\omega / d\omega = n^3\omega^2 / \pi^2 c^3 \ (m^{-3}s)$$

How many photons occupy each mode? Consider two energy states with energies of E_1 and E_2. The probabilities of finding the system in states E_1 and E_2 relate as $P(E_1)/P(E_2) = e^{-(E_1-E_2)/k_BT}$. The state with n_ω photons has energy $n_\omega \hbar\omega$. The mean (expected) number of photons is then

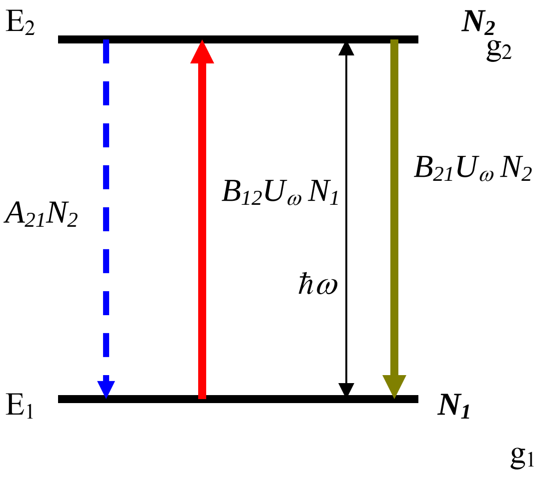
$$\langle n_\omega \rangle = \frac{\sum_{n_\omega=0}^{\infty} n_\omega P(n_\omega)}{\sum_{n_\omega=0}^{\infty} P(n_\omega)} = \frac{\sum_{n_\omega=0}^{\infty} n_\omega e^{-n_\omega \hbar\omega/k_BT}}{\sum_{n_\omega=0}^{\infty} e^{-n_\omega \hbar\omega/k_BT}} = \frac{\sum_{n_\omega=0}^{\infty} \frac{d}{dx} e^{-n_\omega x}}{\sum_{n_\omega=0}^{\infty} e^{-n_\omega x}} = \frac{\frac{d}{dx}\left(\frac{1}{1-e^{-x}}\right)}{1/(1-e^{-x})} = \frac{1}{e^x - 1} = \frac{1}{e^{\hbar\omega/k_BT} - 1}$$

Therefore, the energy density of EM radiation, i.e. energy per unit interval of frequencies per unit volume is

$$U_{\omega,0} = \hbar\omega \langle n_\omega \rangle \rho(\omega) = \frac{n^3\omega^2}{\pi^2 c^3} \frac{\hbar\omega}{e^{\hbar\omega/k_BT} - 1} \ (J \cdot s / m^3)$$

This is a Plank's formula for Black Body Radiation

Einstein's coefficients



g–degeneracy

Balance equation:

$$\frac{dN_2}{dt} = -\frac{dN_1}{dt} = B_{12}(\omega)U_\omega N_1 - A_{21}N_2 - B_{21}(\omega)U_\omega N_2$$

Units of B_{21} are $m^3 / J \cdot s^2$

At equilibrium

$$\frac{N_2}{N_1}=\frac{B_{12}(\omega)U_{\omega,0}}{B_{21}(\omega)U_{\omega}+A_{21}(\omega)}=\frac{B_{12}/B_{21}}{1+A_{21}/B_{21}U_{\omega,0}}=$$

$$=\frac{B_{12}/B_{21}}{1+A_{21}(e^{\hbar\omega/k_BT}-1)/\left(B_{21}\hbar\omega\rho(\omega)\right)}A=\frac{g_2}{g_1}e^{-\hbar\omega/k_BT}$$

Therefore

$$g_1B_{12}(\omega)=g_2B_{21}(\omega)$$

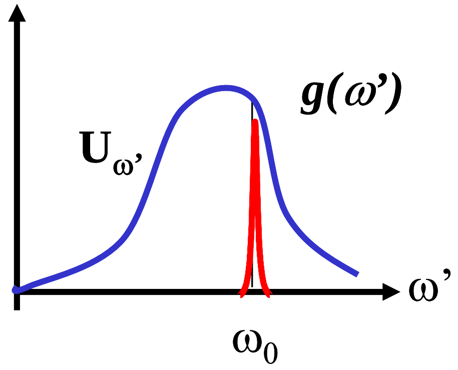
$$A_{21}(\omega)=B_{21}(\omega)\hbar\omega\rho(\omega)=B_{21}(\omega)\hbar\omega\frac{\omega^2n^3}{\pi^2c^3}$$

Introduce broadening g(ω)

Balance Equation in the presence of external broadband radiation U_ω:

$$\frac{dN_2}{dt}=-\frac{dN_1}{dt}=B_{12}N_1\int g(\omega')U_{\omega'}d\omega' \;-B_{21}N_2\int g(\omega')U_{\omega'}d\omega'-A_{21}N\int g(\omega')d\omega$$

Assume first that the line broadening is very small i.e. $g(\omega')=\delta(\omega'-\omega_0)$ then

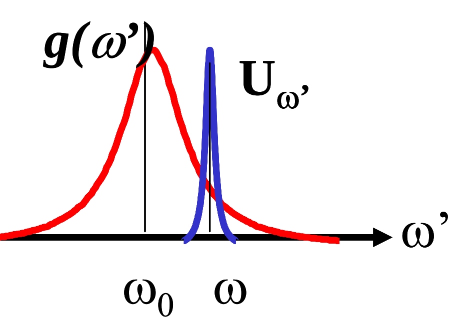


$$\frac{dN_2}{dt}=-\frac{dN_1}{dt}=B_{12}U_{\omega_0}N_1-A_{21}N_2-B_{21}U_{\omega_0}N_2\,,$$

which is, essentially an original result.

Assume now that U_ω is energy density of an incoming monochromatic wave – opposite extreme

$$U_{\omega'}=U\delta(\omega'-\omega)=\frac{n}{c}S\delta(\omega'-\omega)$$



$$\frac{dN_2}{dt}=-\frac{dN_1}{dt}=B_{12}g(\omega)\frac{n}{c}SN_1-B_{21}g(\omega)\frac{n}{c}SN_2-A_{21}N_2$$

If N_2=0;

The energy change per unit volume due to absorption

$$\frac{dU_a}{dt} = \hbar\omega\frac{dN_2}{dt} = \hbar\omega B_{12}g(\omega)\frac{n}{c}SN = -\frac{dS}{dz}$$

So $\frac{dS}{dz} = -N\hbar\omega B_{12}g(\omega)\frac{n}{c}S = -\alpha S = -N\sigma_{12}(\omega)S$

The absorption cross-section is

$$\sigma_{12}(\omega) = \hbar\omega B_{12}g(\omega)\frac{n}{c} = \frac{g_2}{g_1}\frac{\pi^2c^2}{\omega^2n^2}A_{21}g(\omega)$$

Now, we can re-write this as a product of the stimulated emission cross-section and spontaneous radiative lifetime $\tau_{sp} = A_{21}^{-1}$

$$\sigma_{12}(\omega)\tau_{sp} = \frac{g_2}{g_1}\frac{\pi^2c^2}{\omega^2n^2}g(\omega) = \frac{1}{4}\frac{g_2}{g_1}\lambda^2g(\omega)$$

This looks very much like the classical relation

$$\sigma(\omega)\tau_R = \frac{3\pi^2c^2}{n^2\omega^2}g(\omega) = \frac{3}{4}\lambda^2g(\omega)$$

except for the factor of 1/3 (due to the averaging over three directions) and the degeneracy factor.

Introduce oscillator strength as

$$\gamma_R f_{12} = \frac{1}{3}A_{21}$$

$$f_{12} = \frac{1}{3}\frac{g_2}{g_1}\frac{A_{21}}{\gamma_R} = \frac{2\pi\varepsilon_0mc^3}{ne^2\omega^2}\frac{g_2}{g_1}A_{21}$$

Then we obtain for the absorption cross-section

$$\sigma_{12}(\omega) = \frac{\pi e^2}{2cn\varepsilon_0m}f_{12}g(\omega)$$

The only difference between the quantum theory and the classical oscillator model is the presence of oscillator strength!

Now if we have population on the second level we can also introduce the stimulated emission cross-section as

$$\sigma_{21}(\omega) = \frac{\pi e^2}{2cn\varepsilon_0 m} f_{21} g(\omega) = \frac{g_1}{g_2}\sigma_{12}(\omega)$$

The correct expression for susceptibility then becomes:

$$\chi(\omega) = \sum_{m,n>m} \frac{e^2}{\varepsilon_0 m} \frac{N_m - \frac{g_m}{g_n} N_n}{\omega_{mn}^2 - \omega^2 - j\omega\gamma} f_{mn}$$

Let us now write a system of rate equation for the densities of atoms on different levels and the photon flux $S_p = S / \hbar\omega$.

$$\frac{dN_2}{dt} = -\frac{dN_1}{dt} = -\sigma_{21} S_p \left(N_2 - \frac{g_2}{g_1} N_1 \right) - A_{21} N_2 - \frac{N_2}{\tau_{nonrad}}$$

$$\frac{dS_p}{dz} = \sigma_{21} S_p \left(N_2 - \frac{g_2}{g_1} N_1 \right)$$

where

$$\sigma_{21}(\omega) = \frac{\pi^2 c^2}{\omega^2 n^2} A_{21} g(\omega)$$

The condition for gain is **population inversion:**

$$N_2 - \frac{g_2}{g_1} N_1 > 0$$