Lecture 10 Gaussian Beams

Consider the wave equation in the free space

$\nabla^2 E(x,y,z,t) - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}E(x,y,z,t) = 0;$

Consider the "slow variable envelope" approach, i.e. assume that

$E(x,y,z,t) = A(x,y,z)e^{j(k_z z-\omega t)}$

where A is complex envelope, $k_z=n\omega/c$ and the "slow envelope" means that the envelope amplitude A does not change much over the wavelength, i.e. $\partial^2 A/\partial z^2 << k_z \partial A/\partial z$. Then we obtain

$$-k_z^2 A + \frac{\partial^2 A}{\partial z^2} + 2jk_z\frac{\partial A}{\partial z} + \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{n^2}{c^2}\omega^2 A = 0$$

or

$$\frac{\partial A(x,y,z)}{\partial z} = \frac{j}{2k_z}\nabla_t^2 A(x,y,z); \quad \nabla_t^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

This equation describes the phenomenon of diffraction – in-plane variations of the amplitude cause the change of both amplitude and phase along the direction of propagation.

Let us introduce a 2-Dimensional spatial Fourier transform, i.e. express the wave E as a sum of plane waves propagating at different angles to the axis z.

$$A(z;x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{A}(z;k_x,k_y)e^{j(k_x x+k_y y)}dk_x dk_y$$

where

$$\tilde{A}(z;k_x,k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A(z;x,y)e^{-j(k_x x+k_y y)}dxdy$$

Apply Fourier transform to our wave equation. Easy to see that derivative $\partial A/\partial x$ is transformed as $jk_x\tilde{A}$. Therefore we obtain

$$\frac{\partial}{\partial z}\tilde{A}(z;k_x,k_y) = -\frac{k_x^2+k_y^2}{2k_z}\tilde{A}(z;k_x,k_y),$$

whose solution is

$$\tilde{A}(z;k_x,k_y) = \tilde{A}(0;k_x,k_y)e^{-j\frac{k_x^2+k_y^2}{2k_z}z}$$

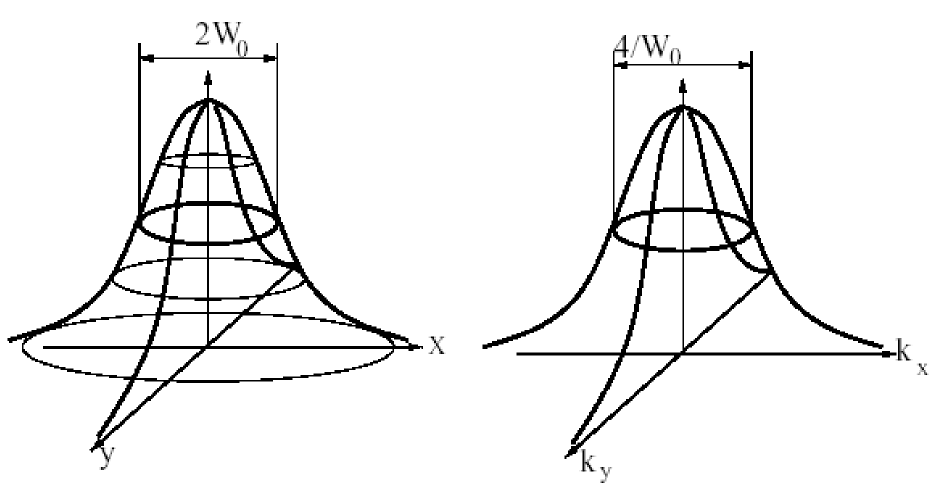
i.e. the initial beam profile in Fourier space multiplied by a Gaussian phase factor.

Assume now that the initial beam profile is Gaussian,

$$A(0;x,y) = A_0 e^{-\frac{x^2+y^2}{w_0^2}} = A_0 e^{-\frac{r^2}{w_0^2}}$$

where the w_0 is the beam waist radius and amplitude at the center is related to the beam total power as

$$P_0 = \frac{1}{2\eta}\int\int |A(0;x,y)|^2\,dxdy = \frac{A_0^2}{2\eta}2\pi\int e^{-\frac{2r^2}{w_0^2}}rdr = \frac{\pi}{4\eta}A_0^2 w_0^2$$



The Fourier transform of the Gaussian beam is also Gaussian

$$\tilde{A}(0;k_x,k_y)=A_0\int\int e^{-\frac{x^2+y^2}{w_0^2}-j(k_x x+k_y y)}dxdy=A_0\int e^{-\frac{x^2}{w_0^2}-jk_x x+\frac{k_x^2w_0^2}{4}-\frac{k_x^2w_0^2}{4}}dx\int e^{-\frac{y^2}{w_0^2}-jk_y x+\frac{k_y^2w_0^2}{4}-\frac{k_y^2w_0^2}{4}}dy=$$

$$A_0e^{-\frac{k_x^2+k_y^2}{4}w_0^2}\int e^{-\left(\frac{x}{w_0}+\frac{jk_xw_0}{2}\right)^2}dx\int e^{-\left(\frac{y}{w_0}+\frac{jk_yw_0}{2}\right)^2}dy=\pi A_0w_0^2e^{-\frac{k_x^2+k_y^2}{4}w_0^2}$$

,

with the waist radius in Fourier space equal to $2/w_0$. Therefore the Fourier transform at distance z is

$$\tilde{A}(z;k_x,k_y)=\tilde{A}(0;k_x,k_y)e^{-j\frac{k_x^2+k_y^2}{2k_z}z}=\pi A_0w_0^2e^{-\left(k_x^2+k_y^2\right)\left(\frac{w_0^2}{4}+j\frac{z}{2k_z}\right)}=\pi A_0w_0^2e^{-\left(k_x^2+k_y^2\right)\frac{w_0^2}{4}\left(1+j\frac{z}{z_0}\right)}$$

where we introduced the **diffraction length** $z_0=k_zw_0^2/2=\pi w_0^2n/\lambda$

Now let us do inverse Fourier transform

$$A(z;x,y)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{A}(z;k_x,k_y)e^{j(k_xx+k_yy)}dk_xdk_y=\frac{A_0w_0^2}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\left(k_x^2+k_y^2\right)\frac{w_0^2}{4}\left(1+j\frac{z}{z_0}\right)+j(k_xx+k_yy)}dk_xdk_y$$

Transform the expression in the exponent:

$$-\left(1+j\frac{z}{z_0}\right)\frac{w_0^2}{4}\left(k_x^2+k_y^2-2j\frac{2}{w_0^2}\frac{k_xx+k_yy}{1+jz/z_0}\right)=$$

$$=-\left(1+j\frac{z}{z_0}\right)\frac{w_0^2}{4}\left[\left(k_x-j\frac{2k_xx}{w_0^2(1+jz/z_0)}\right)^2+\left(k_y-j\frac{2k_yx}{w_0^2(1+jz/z_0)}\right)^2-4\frac{x^2+y^2}{w_0^4(1+jz/z_0)^2}\right]=$$

$$=-\left(1+j\frac{z}{z_0}\right)\frac{w_0^2}{4}\left[\left(k_x-j\frac{2k_xx}{w_0^2(1+jz/z_0)}\right)^2+\left(k_y-j\frac{2k_yx}{w_0^2(1+jz/z_0)}\right)^2\right]-\frac{x^2+y^2}{w_0^2(1+jz/z_0)}$$

Therefore

$$A(z;x,y)=\frac{A_0}{1+jz/z_0}e^{-\frac{x^2+y^2}{w_0^2(1+jz/z_0)}}=\frac{A_0}{\sqrt{1+\left(z/z_0\right)^2}}e^{-\frac{x^2+y^2}{w_0^2\left[1+\left(z/z_0\right)^2\right]}(1-jz/z_0)-j\tan^{-1}\frac{z}{z_0}}=$$

$$=\frac{A_0w_0}{w(z)}e^{-\frac{x^2+y^2}{w^2(z)}}e^{jk\frac{x^2+y^2}{2R(r)}-j\eta(z)}$$

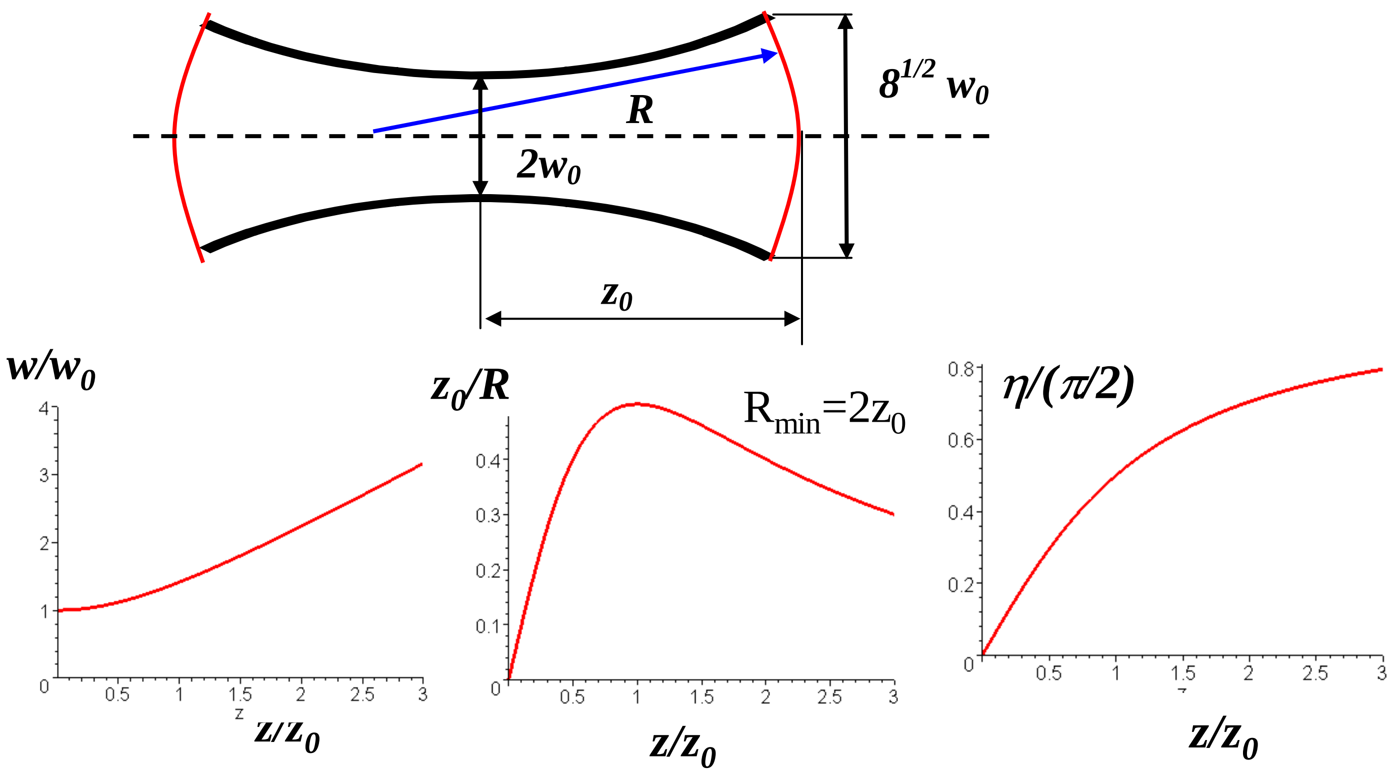
where the beam radius at point z is

$$w(z) = w_0(1 + z^2/z_0^2)^{1/2}$$

radius of curvature at point z is

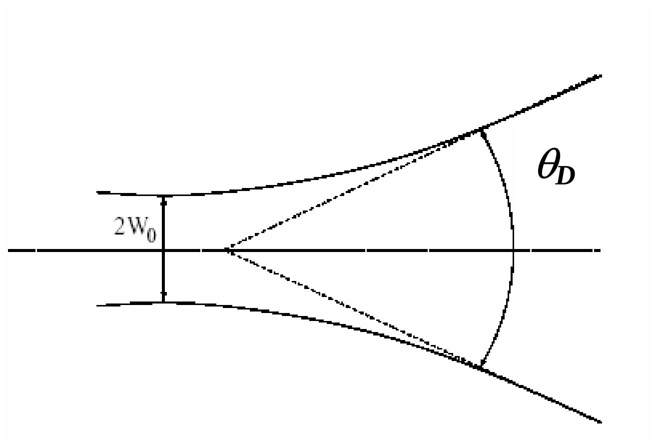
$$R(z) = \left[1 + (z/z_0)^2\right]\frac{\pi n}{\lambda}w_0^2\frac{z_0}{z} = \frac{z_0^2}{z}\left[1 + (z/z_0)^2\right] = z\left[1 + (z_0/z)^2\right]$$

and additional phase shift is $\eta(z)=\tan^{-1}(z/z_0)$



So, at one diffraction length the beam radius increases by a square root of two, thus the peak intensity is reduced by a factor of two.

For large distances $z>>z_0$ $2w(z)\approx 2zw_0/z_0=2z\lambda/n\pi w_0=z\theta_d$, where diffraction angle is $\theta_D=2\lambda/n\pi w_0$

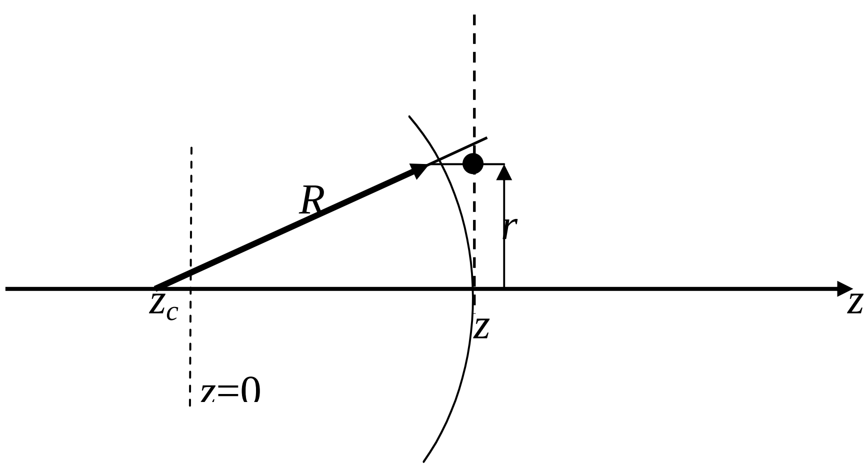


The phase term can be written as

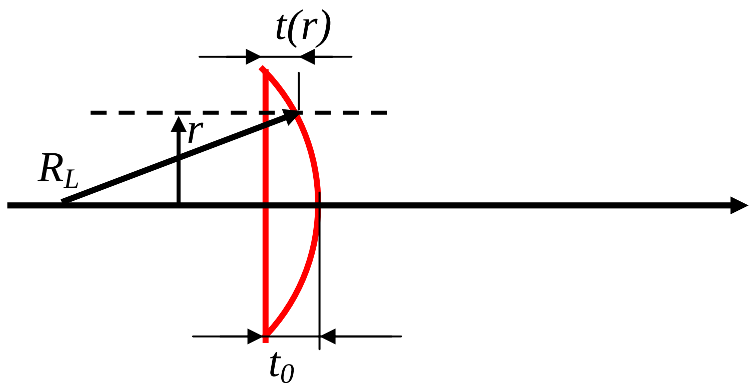
$$\varphi(z,x,y)=k(z+\frac{x^2+y^2}{2R})-\eta(z)=k\left(R+\frac{x^2+y^2}{2R}\right)+k\left[z-R-k^{-1}\eta(z)\right]$$

$$\approx k\sqrt{R^2+x^2+y^2}+kz_c;$$

where $z_c=z-R-k^{-1}\eta(z)\approx -z_0^2/z$



Gaussian beam propagation through the thin lens



$$t(r) = t_0 - \left(R_L - \sqrt{R_L^2 - r^2}\right) \approx t_0 - r^2/2R_L$$

Phase shift in the lens is

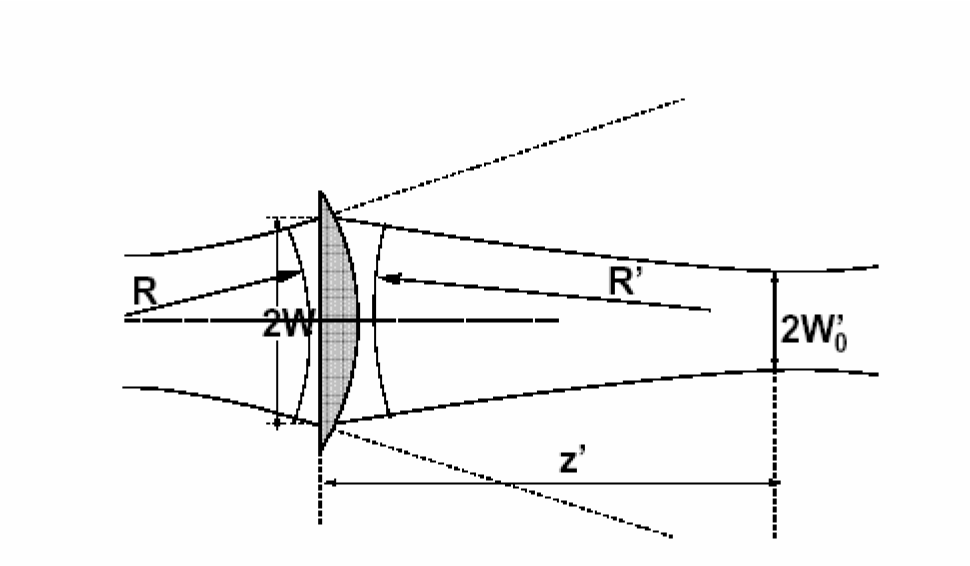
$$\delta\varphi(r) = k_0 n t(r) + k_0\left[t_0 - t(r)\right] = k_0 t_0 + (n-1)k_0 t(r) = \varphi_0 - k_0\frac{n-1}{2R_L}r^2 = \varphi_0 - k_0\frac{r^2}{2f}$$

where $f=R_L/(n-1)$ is the focal length.

Therefore we can write for the phase of the Gaussian beam following the lens:

$$\varphi_+(z,x,y) = \varphi_-(z,x,y) + \delta\varphi(x,y) =$$
$$k_0(z + \frac{x^2+y^2}{2R}) - \eta(z) + \varphi_0 - k_0\frac{x^2+y^2}{2f} = \varphi(z) + \frac{x^2+y^2}{2R'}$$

where the new radius of curvature is $\frac{1}{R'} = \frac{1}{R} - \frac{1}{f}$. If $R'<0$, the beam starts converging.



Matrices and Gaussian beams

$$A(z;x,y)=\frac{A_0}{1+jz/z_0}e^{-\frac{x^2+y^2}{w_0^2(1+jz/z_0)}}=$$

$$=\frac{A_0}{1+jz/z_0}e^{-k_z\frac{x^2+y^2}{2z_0(1+jz/z_0)}}=\frac{-jz_0A_0}{z-jz_0}e^{jk_z\frac{x^2+y^2}{2(z-jz_0)}}=\frac{-jz_0A_0}{q}e^{jk_z\frac{x^2+y^2}{2q}}$$

Now parameter $q=z-jz_0$ incorporates both spot size and radius of curvature of the Gaussian mode.

$$\frac{1}{q}=\frac{z+jz_0}{z^2+z_0^2}=\frac{1}{z\left[1+(z_0/z)^2\right]}+j\frac{1}{z_0\left[1+(z/z_0)^2\right]}=\frac{1}{R}+j\frac{\lambda}{\pi n w^2}$$

Now consider the propagation through the length d of free space with matrix

$T_d=\begin{vmatrix}1 & d\\ 0 & 1\end{vmatrix}$. Easy to see that $q_2=q_1+z=\frac{Aq_1+B}{Cq_1+D}$

Next look at propagation through the thin lens $L_f=\begin{vmatrix}1 & 0\\ -1/f & 1\end{vmatrix}$

Easy to see that $\frac{1}{q_2}=\frac{1}{R_2}+j\frac{\lambda}{\pi w_2^2}=\frac{1}{R_1}-\frac{1}{f}+j\frac{\lambda}{\pi w_1^2}=\frac{1}{q_1}-\frac{1}{f}$

Then $q_2=\frac{q_1}{1-q_1/f}=\frac{Aq_1+B}{Cq_1+D}$. Therefore, for arbitrary matrix we still have

$$q_2=\frac{Aq_1+B}{Cq_1+D}$$

Gaussian mode in a resonator

If the round trip in cavity is described by an ABCD matrix then the mode must reproduce itself after one round trip, i.e.

$$q_2=\frac{Aq_1+B}{Cq_1+D}=q_1\equiv q;\ Cq^2-(A-D)q-B=0\ ;$$

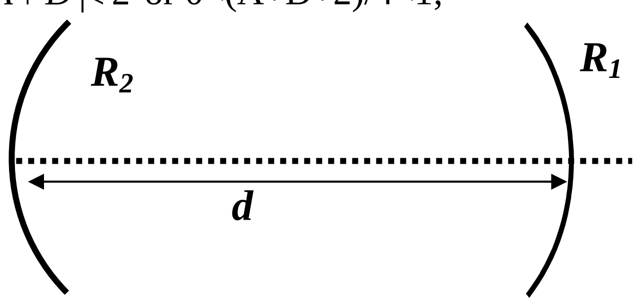
$$B\frac{1}{q^2}+(A-D)\frac{1}{q}-C=0;$$

$$\frac{1}{q}=-\frac{A-D}{2B}\pm\frac{1}{B}\sqrt{\left(\frac{A-D}{2}\right)^2+BC}=-\frac{A-D}{2B}\pm\frac{1}{B}\sqrt{\left(\frac{A-D}{2}\right)^2+AD-1}=$$

$$=-\frac{A-D}{2B}\pm\frac{j}{B}\sqrt{1-\left(\frac{A+D}{2}\right)^2}=\frac{1}{R}+j\frac{\lambda}{\pi n w^2}$$

The waist radius must be real, therefore stability condition is once again

$|A+D|<2$ or 0<(A+D+2)/4<1;



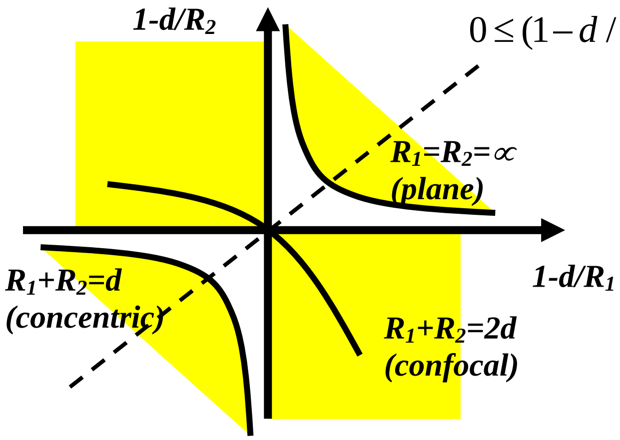
Now, the matrix itself is $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} 1-\frac{2d}{R_2} & d+d\left(1-\frac{2d}{R_2}\right) \\ -\frac{2}{R_1}-\frac{2}{R_1}\left(1-\frac{2d}{R_1}\right) & \left(1-\frac{2d}{R_1}\right)\left(1-\frac{2d}{R_2}\right)-\frac{2d}{R_1} \end{vmatrix}$

For the cavity with two spherical mirrors one obtains the same stability condition as obtained using the ray tracing,

$$\frac{A+D+2}{4} = \frac{1}{4}\left[\left(1-\frac{2d}{R_2}\right)+\left(1-\frac{2d}{R_1}\right)\left(1-\frac{2d}{R_2}\right)-\frac{2d}{R_1}+2\right]=$$

$$=\frac{1}{4}\left[4-\frac{4d}{R_1}-\frac{4d}{R_2}+\frac{4d^2}{R_1R_2}\right]=\left(1-\frac{d}{R_1}\right)\left(1-\frac{d}{R_2}\right)$$

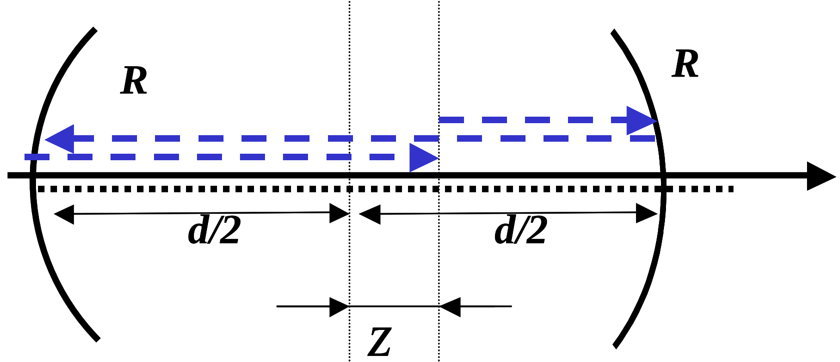
$$0 \le (1-d/R_1)(1-d/R_2) \le 1$$



If the cavity is symmetric we obtain

$$-1 \le (1-d/R) \le 1 \;\text{ or }\; 1\text{-}d/2R \le 1$$

Let us now find out the waist radius for the symmetric cavity. The ABCD matrix, starting from some distance z from the center is



$$\begin{bmatrix} 1 & \frac{d}{2}+z \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2/R & 1 \end{bmatrix}\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2/R & 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{d}{2}-z \\ 0 & 1 \end{bmatrix}=$$

$$\begin{bmatrix} 1-4\frac{d}{R}+2\frac{d^2}{R^2}-4z\left(\frac{1}{R}-\frac{d}{R^2}\right) & d\left(\frac{d^2}{R^2}-3\frac{d}{R}+2\right)+4z^2\left(\frac{1}{R}-\frac{d}{R^2}\right) \\ -4\left(\frac{1}{R}-\frac{d}{R^2}\right) & 1-4\frac{d}{R}+2\frac{d^2}{R^2}+4z\left(\frac{1}{R}-\frac{d}{R^2}\right) \end{bmatrix}$$

Let us find the radius of curvature

$$\frac{1}{R_c(z)}=-\frac{A-D}{2B}=\frac{4z}{B}\left(\frac{1}{R}-\frac{d}{R^2}\right)$$

So, at $z=0$ $1/R_c=0$ –as expected from the symmetry the waist is at the center of cavity. Note that at $z=\pm d/2$, $B=2d(1-d/R)$ and we obtain $1/R_c=\pm 1/R$ as demanded.

Now, let us find the beam waist radius at some point z

$$w^2(z)=\frac{|B|\lambda}{\pi n}\left[1-\left(\frac{A+D}{2}\right)^2\right]^{-1/2}=\frac{\lambda d}{\pi n}\frac{|\left(2-d/R\right)(1-d/R)+4\left(z^2/d^2\right)\left(d/R\right)(1-d/R)|}{2\sqrt{\left(d/R\right)(2-d/R)(1-d/R)^2}}=$$

$$\frac{\lambda d}{\pi n}\frac{|\left(2-d/R\right)+4\left(z^2/d^2\right)\left(d/R\right)|}{2\sqrt{\left(d/R\right)(2-d/R)}}=$$

At the beam waist we obtain

$$w^2(0)=\frac{\lambda d}{\pi n}\frac{\sqrt{2-d/R}}{2\sqrt{\left(d/R\right)}}=\frac{\lambda d}{2\pi n}\sqrt{2R/d-1}$$

At the mirror we obtain

$$w^2(d/2)=\frac{\lambda d}{\pi n}\frac{1}{\sqrt{\left(d/R\right)(2-d/R)}}=w_0^2\frac{2}{2-d/R}$$

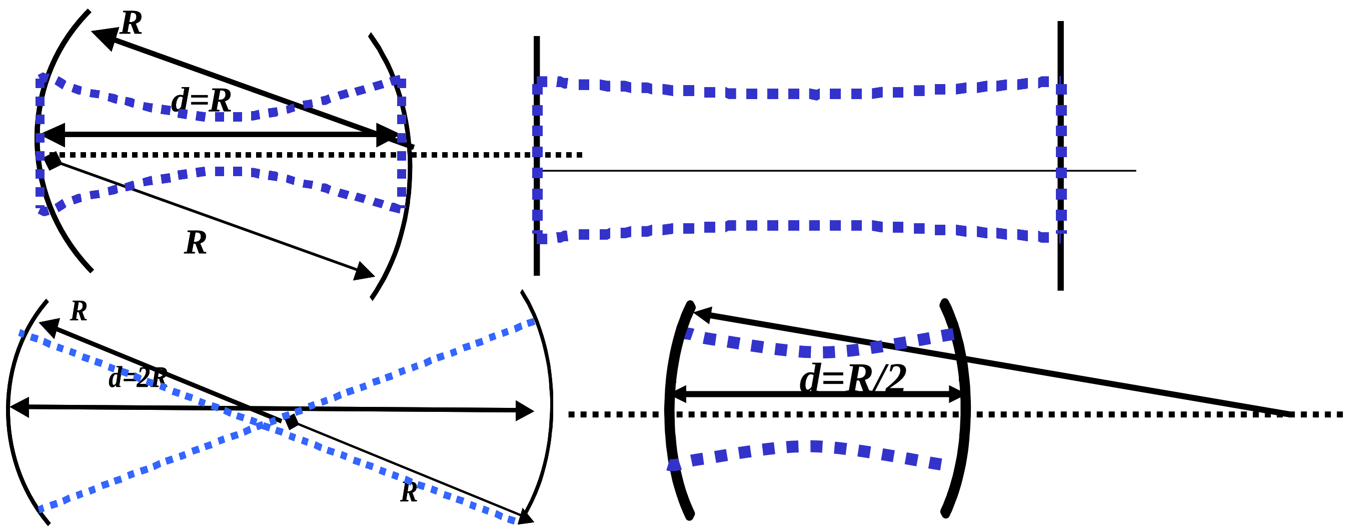
Consider now some specific cavities:

Symmetric confocal cavity –$d=R$ $w_0^2=\dfrac{\lambda d}{2\pi n}$; $w^2(d/2)=\dfrac{\lambda d}{\pi n}=2w_0^2$

flat mirror cavity $R\to\infty$; $w_0^2\to\infty$; $w^2(d/2)\approx w_0^2$

Symmetric concentric cavity $d=2R$ $w_0^2\to 0$; $w^2(d/2)\to\infty$

These cavities are too close to the region of instability – for stability, let us choose $d=R/2$; $w_0^2=\dfrac{\lambda d\sqrt{3}}{2\pi n}$; $w^2(d/2)=\dfrac{2\lambda d}{\sqrt{3}\pi n}$



Higher order modes.

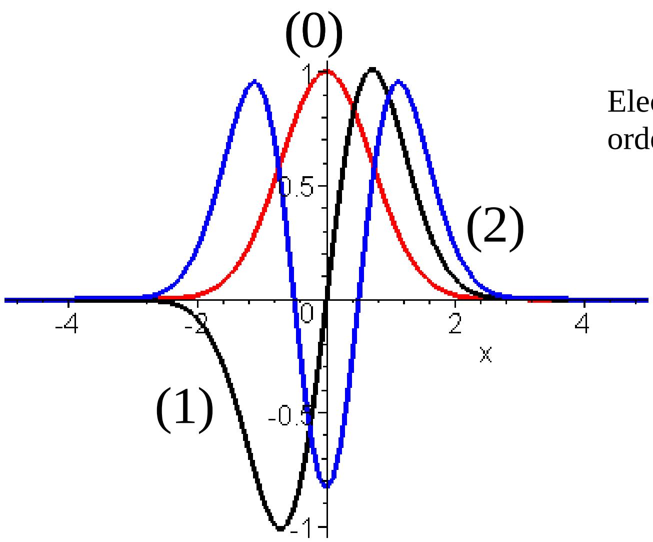
In addition to the fundamental Gaussian solution of wave equation there exist higher order modes, called Gaussian-Hermite

$$A_{p,q}(z,x,y)=H_p\left[\frac{\sqrt{2}x}{w(z)}\right]H_q\left[\frac{\sqrt{2}y}{w(z)}\right]\frac{A_{0,p,q}w_0}{w(z)}e^{-\frac{x^2+y^2}{w^2(z)}}e^{jk\frac{x^2+y^2}{2R(r)}}e^{-j(1+p+q)\eta(z)}$$

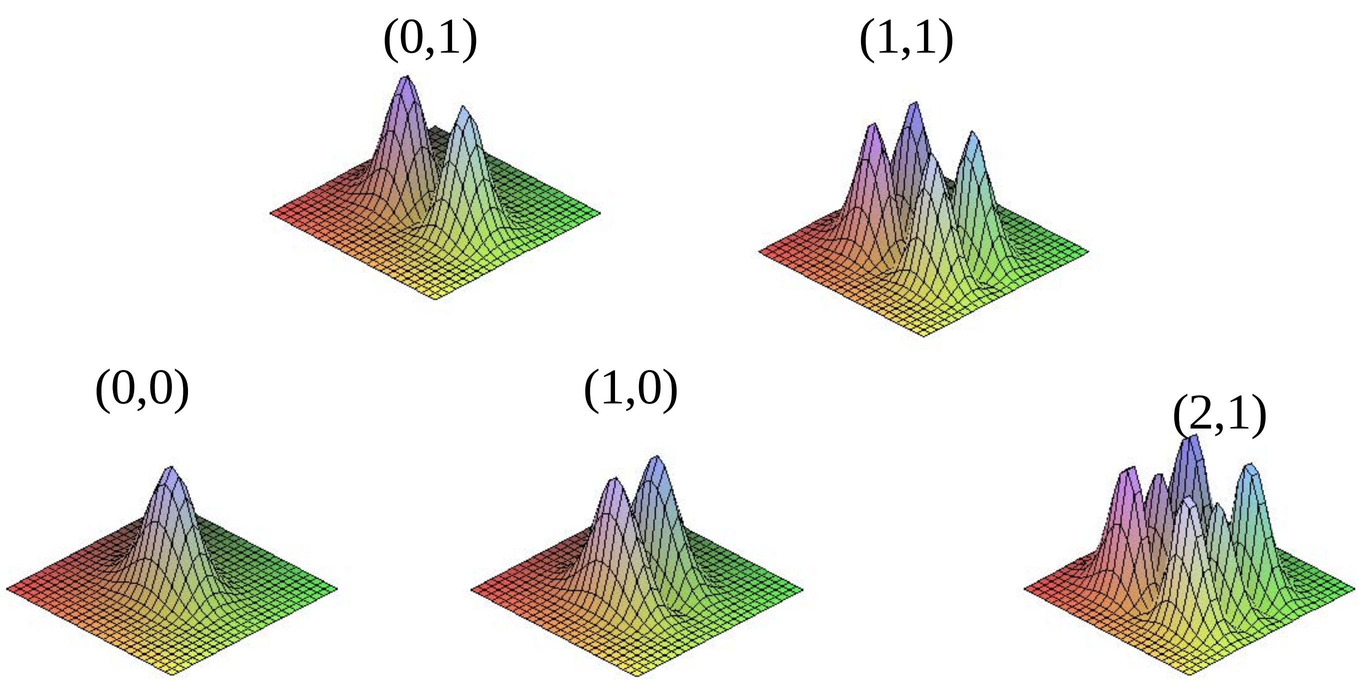
where Hermite polynomials are

$$H_p(u) = (-1)^p e^{u^2} \frac{d^p e^{-u^2}}{du^p};$$
$$H_0(u) = 1;\ \ H_1(u) = 2u;\ \ H_0(u) = 4u^2 - 2;$$

Note that w(z) is the same for all the modes but due to presence of polynomial the spot size of higher order modes is larger.



Electric fields in higher order modes



Mode intensities

Resonant frequency

$$E_{p,q}(z,x,y) = H_p\left[\frac{\sqrt{2}x}{w(z)}\right] H_q\left[\frac{\sqrt{2}y}{w(z)}\right] \frac{A_{0,p,q}w_0}{w(z)} e^{-\frac{x^2+y^2}{w^2(z)}} e^{jk\frac{x^2+y^2}{2R(r)}} e^{k_z z - j(1+p+q)\eta(z)}$$

$$\eta = \tan^{-1}(z/z_0)$$

The resonant occurs when the round trip phase shift is equal to 2mπ. For the symmetric cavity we can write

$$\Delta\varphi_{rt} = 4[\varphi(d/2) - \varphi(0)] = 4\left[k\frac{d}{2} - (1+p+q)\tan^{-1}\left(\frac{d}{2z_0}\right)\right] = 2m\pi$$

But $z_0 = \frac{d}{2}\sqrt{2R/d - 1}$ therefore

$\frac{\pi n \nu_{m,p,q} d}{c} - (1+p+q)\tan^{-1}\left[\left(2R/d - 1\right)^{-1/2}\right] = m\frac{\pi}{2}$; Use the identity $\cos^2 x = 1/(1+\tan^2 x)$ to obtain

$$\nu_{m,p,q} = \frac{c}{2nd}\left[m + 2\frac{1+p+q}{\pi}\cos^{-1}\left(1 - \frac{d}{2R}\right)^{1/2}\right]$$

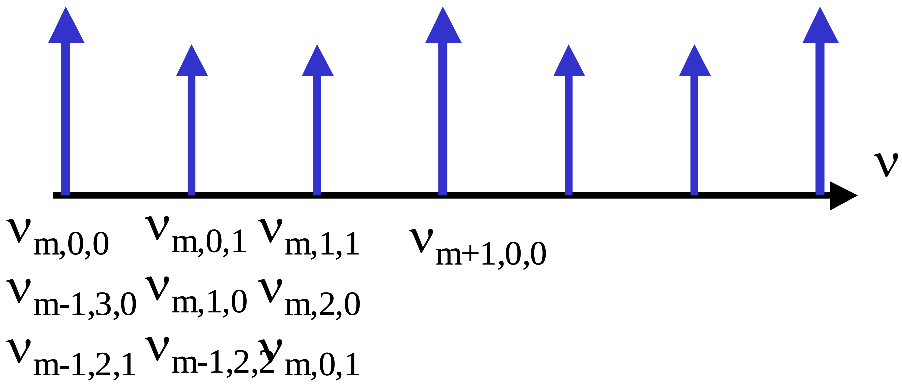
Consider now higher order modes in the specific resonators:

Plane cavity *$R \to \infty$* $\nu_{m,p,q} = \frac{c}{2nd} m$ (modes are degenerate as they should be in Fabry-Perot cavity)

Symmetric confocal cavity d=R $\nu_{m,p,q} = \frac{c}{2nd}\left[m + \frac{1}{2}(p+q+1)\right]$

Symmetric concentric cavity *d=2R* $\nu_{m,p,q} = \frac{c}{2nd}\left[m + p + q + 1\right]$

More typical cavity *d=R/2* $\nu_{m,p,q} = \frac{c}{2nd}\left[m + \frac{1}{3}(p+q+1)\right]$



Diffraction Losses – they depend on the Fresnel number

$$N = \frac{a^2}{\lambda d}$$

This makes sense since the the spotsize at the mirror is of the order of $\pi w^2 \sim \lambda d$

Higher modes have higher losses.