1. Consider two plates of thickness t and refractive indices n and n+Δn. What is the minimum difference Δn can you measure using interference method? Use uncertainty principle and relate refractive index to the momentum.

The uncertainty of index measurement should be less than Δn - meaning that uncertainty of momentum should be $\Delta p_z = \Delta(\hbar k_z) = \frac{2\pi}{\lambda_0}\hbar\Delta n$. The uncertainty of co-ordinate is $\Delta z = t$. Then we can write the uncertainty relation as $\Delta p_z \Delta z = \frac{2\pi}{\lambda_0}\hbar\Delta n t \geq \hbar/2$ or $\Delta n \geq \lambda_0/4\pi t$.

The other way to see it is to find the phase difference that one can obtain by putting plates into separate arms of interferometer. $\delta\varphi = \frac{2\pi}{\lambda_0}\Delta n t$. Assuming that we can see the phase difference of 1/2 rad we obtain the same result.

2. Consider the tunneling through the barrier problem. Calculate the probability current density in three regions and show that the probability current is continuous.

In the region III we have

$$\Psi_{III} = Fe^{jk_1(z-a)}$$

And

$$\mathbf{J}_{p,III} = \frac{\hbar}{m_0}\operatorname{Im}\left(\Psi_{III}^*\nabla\Psi_{III}\right) = F^2\frac{\hbar k_1}{m_0}$$

In the region II

$$\Psi_{II}^* = C^*e^{-q_2 z} + D^*e^{q_2 z}$$

$$\nabla\Psi_{II} = -q_2Ce^{-q_2 z} + q_2De^{q_2 z}$$

$$\mathbf{J}_{p,I} = \frac{\hbar}{m_0}\operatorname{Im}\left(\Psi_I^*\nabla\Psi_I\right) = \frac{\hbar q_2}{m_0}\operatorname{Im}\left(\underbrace{-CC^*e^{-2q_2 z}}_{\text{Real}} + \underbrace{DD^*e^{2q_2 z}}_{\text{Real}} + C^*D - CD^*\right) = \frac{\hbar q_2}{m_0}\operatorname{Im}\left(C^*D - CD^*\right)$$

Then since

$$C = \frac{1}{2}e^{q_2 a}\left(1 - j\frac{k_1}{q_2}\right)F$$

$$D = \frac{1}{2}e^{-q_2 a}\left(1 + j\frac{k_1}{q_2}\right)F$$

$$\mathbf{J}_{p,II} = \frac{\hbar q_2}{m_0}\frac{F^2}{4}\operatorname{Im}\left\{\left(1+j\frac{k_1}{q_2}\right)^2 - \left(1-j\frac{k_1}{q_2}\right)^2\right\} = \frac{\hbar k_1}{m_0}F^2 = \mathbf{J}_{p,III}$$

In region I

$$\Psi_I = Ae^{jk_1z} + Be^{-jk_1z};\ \Psi_I^* = A^*e^{-jk_1z} + B^*e^{jk_1z}$$

$$\nabla\Psi_I = jk_1Ae^{jk_1z} - jk_1Be^{-jk_1z}$$

$$\mathbf{J}_{p,I} = \frac{\hbar}{m_0}\operatorname{Im}\left(\Psi_I^*\nabla\Psi_I\right) = \frac{\hbar k_1}{m_0}\left(AA^* - BB^*\right) =$$

$$= \frac{\hbar k_1}{m_0}F^2\left[\cosh^2(q_2a) + \frac{1}{4}\left(\frac{q_2}{k_1} - \frac{k_1}{q_2}\right)^2\sinh^2(q_2a) - \frac{1}{4}\left(\frac{q_2}{k_1} - \frac{k_1}{q_2}\right)^2\sinh^2(q_2a)\right] =$$

$$\frac{\hbar k_1}{m_0}F^2\left[\cosh^2(q_2a) - \sinh^2(q_2a)\right] = \frac{\hbar k_1}{m_0}F^2 = \mathbf{J}_{p,III}$$