**SOLUTION: FINAL 2008**

**1. What is the dielectric constant of the ideal (no resistance) metal and what is the physical meaning of it?**

The dielectric constant of ideal metal is

$$\varepsilon = 1 - \frac{\omega_p^2}{\omega^2}$$

For frequencies less than $\omega_p$ the dielectric constant is negative – it means that the electromagnetic wave cannot propagate in the metal because the electrons move very fast to screen the field. As frequency goes to zero the dielectric constant goes to infinity – which makes sense because the dielectric constant is the measure of polarizability, i.e. ability of charge to move when the field is applied. In the metal the charges are free hence there is nothing to prevent these charges from moving – hence $\varepsilon \to \infty$. When frequency exceeds plasma frequency the dielectric constant becomes positive and the metal behaves just like a normal dielectric because the electrons cannot follow the rapidly changing field.

**2. Why is it difficult to make an X-ray lens or mirror?**

Consider the dielectric constant of any atom with many transitions

$$\varepsilon = 1 + \sum_i \frac{\omega_{pi}^2}{\omega_{0i}^2 - \omega^2 - j\omega\gamma}$$

As frequency becomes very high eventually

$$\lim_{\omega\to\infty} \varepsilon = 1 - \sum_i \frac{\omega_{pi}^2}{\omega^2} = 1 - \frac{\sum_i \omega_{pi}^2}{\omega^2} \to 1$$

Therefore index of refraction n~1 fro all the materials, and there is almost reflection or refraction anymore. One needs many layers of different materials to get appreciable reflection in X-ray range. The other way to look at it is to see that the momentum of X-rays

$$p = \hbar k = \hbar\omega / c$$

is very large and is difficult to change – hence the X-ray photon just goes straight through most of the materials.

**3. Consider a one-dimensional material "nanowire" with normal parabolic energy-momentum relationship. Derive the expression for the density of states. How does the position of Fermi level and electrical conductivity change with increase of electron concentration? Concentration n is per unit length, i.e. per cm.**

Consider a nanowire of length L – according to cyclical boundary conditions the allowed states must have wave vectors $k_m = 2\pi / Lm$ - hence the total number of states with wave-vectors smaller than $k$ is

$$N_k = 2 \times kL / 2\pi$$

Hence the number of states within interval from k to k+dk per unit length is

$$dN_k = dk / \pi$$

But in parabolic band

$$k = \sqrt{\frac{2m^* \mathrm{E}}{\hbar^2}}; \;\; dk = \sqrt{\frac{m^*}{2\hbar^2 \mathrm{E}}} d\mathrm{E}$$

and

$$dN_{\mathrm{E}} = \sqrt{\frac{m^*}{2\pi^2\hbar^2}} \mathrm{E}^{-1/2} d\mathrm{E}$$

Thus the density of states is

$$\rho(\mathrm{E}) = \sqrt{\frac{m^*}{2\pi^2\hbar^2}} \mathrm{E}^{-1/2}$$

To find the concentration of carriers at low temperature we can write

$$n = \int_0^{E_F} \rho(\mathrm{E}) d\mathrm{E} = \sqrt{\frac{m^*}{2\pi^2\hbar^2}} \int_0^{\mathrm{E}_F} \frac{d\mathrm{E}}{\mathrm{E}^{1/2}} = \sqrt{\frac{2m^* \mathrm{E}_F}{\pi^2\hbar^2}}$$

Hence

$$\mathrm{E}_F = \frac{\pi^2\hbar^2 n^2}{2m^*}$$

Dependence is quadratic.
The electrical current is

$$\mathbf{I} = -e^2\tau_s \sum_{\mathbf{k}} \frac{\partial f^0(\mathbf{k})}{\partial \mathrm{E}} \mathbf{v_k} \left( \mathbf{v_k} \cdot \mathbf{E} \right)$$

Note that in one dimension there is no such thing as current density – only the current.
Since we are dealing with only one dimension we get

$$I = -e^2\tau_s \sum_k \frac{\partial f^0(k)}{\partial \mathrm{E}} v^2 E = \sigma E$$

Going from summation over k-vector to integration over energy we get

$$\sigma = -e^2\tau_s \int \frac{\partial f^0(k)}{\partial \mathrm{E}} v^2(\mathrm{E})\rho(\mathrm{E})d\mathrm{E} = e^2\tau_s \int \delta(\mathrm{E}-\mathrm{E}_F)v^2(\mathrm{E})\rho(\mathrm{E})d\mathrm{E} = e^2\tau_s v^2(\mathrm{E_F})\rho(\mathrm{E_F})$$

But in a parabolic band

$$\frac{m^* v^2}{2} = \mathrm{E} \qquad (1)$$

Therefore

$$\sigma = e^2\tau_s \frac{2\mathrm{E_F}}{m^*}\sqrt{\frac{m^*}{2\pi^2\hbar^2}}\mathrm{E}_F^{-1/2} = \frac{e^2\tau_s}{m^*}\sqrt{\frac{2m^*\mathrm{E_F}}{\pi^2\hbar^2}} = \frac{ne^2\tau_s}{m^*} \qquad (2)$$

Which is the same expression as for 3D (and 2D) except the concentration is in 1/cm and the conductivity is accordingly in cm/Ω.

**4. Suppose you have a radiation from two sources coming at you. One has frequency $\nu_1$ and the other $\nu_2$. What is the minimum time it should take to decide from which source the radiation comes?**

You have to measure the difference in energy that is $\Delta\mathrm{E} = \hbar(\nu_2 - \nu_1)$. According to the uncertainty principle it should take $\Delta\tau \sim \hbar / |\Delta\mathrm{E}| \sim 1/|\nu_2 - \nu_1|$