

Solutions to Homework #2

Problem 1

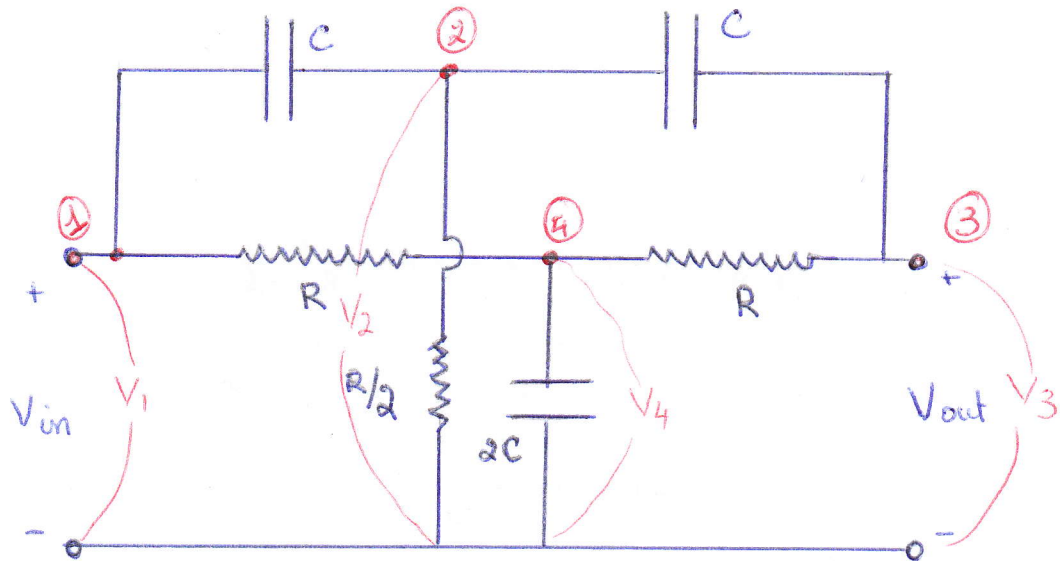


Figure 1: Labeling of nodes for circuit in Problem # 1

- Consider the circuit with nodes labeled as shown in Figure 1. Applying Kirchoff's current law at node 2, we have:

$$C \frac{d(V_2 - V_1)}{dt} + C \frac{d(V_2 - V_3)}{dt} + \frac{2V_2}{R} = 0 \quad (1)$$

Applying Kirchoff's current law at node 3, we have:

$$\frac{1}{R}(V_3 - V_4) = C \frac{d(V_2 - V_3)}{dt} \quad (2)$$

Applying Kirchoff's current law at node 4, we have:

$$\frac{V_4 - V_1}{R} + \frac{V_4 - V_3}{R} + 2C \frac{dV_4}{dt} = 0 \quad (3)$$

The above set of 3 coupled ODEs describe the system with 3 variables, V_2 , V_3 and V_4 . Voltage V_1 , being an input of the system, is given.

- Taking the (unilateral) Laplace transform of (1), (2) and (3) respectively, assuming 0 initial conditions, we have:

$$\left(\frac{2}{R} + 2Cs\right)V_2(s) - CsV_3(s) - CsV_1(s) = 0 \quad (4)$$

$$-CsV_2(s) + \left(\frac{1}{R} + Cs\right)V_3 - \frac{1}{R}V_4 = 0 \quad (5)$$

$$-\frac{1}{R}V_3 + \left(\frac{2}{R} + 2Cs\right)V_4 - \frac{1}{R}V_1(s) = 0 \quad (6)$$

Solving (4) for $V_2(s)$ in terms of $V_1(s)$ and $V_3(s)$, we get:

$$V_2(s) = \frac{RCs}{2(1+RCs)}V_1(s) + \frac{RCs}{2(1+RCs)}V_3(s)$$

Plugging into (5) and solving for $V_4(s)$ in terms of $V_1(s)$ and $V_3(s)$, we get:

$$V_4(s) = -\frac{(RCs)^2}{2(1+RCs)}V_1(s) + \left(1 + RCs - \frac{(RCs)^2}{2(1+RCs)}\right)V_3(s)$$

Finally, plugging this expression into (6) and rearranging the terms we get:

$$V_3(s) = \frac{1 + (RC)^2s^2}{1 + 4RCs + (RC)^2s^2}V_1(s)$$

Thus the system with input V_1 and output V_2 has transfer function:

$$H(s) = \frac{s^2 + \alpha^2}{s^2 + 4\alpha s + \alpha^2}, \text{ where } \alpha = \frac{1}{RC}$$

Remark: Note that we started out with 3 ODEs in the first part, yet the transfer function obtained in this part is a second order transfer function. This is due to pole-zero cancellations.

- Note that $H(s)$ has two real poles in the O.L.H.P. at $(-2 - \sqrt{3})\alpha$ and $(-2 + \sqrt{3})\alpha$ and a pair of complex zeroes on the imaginary axis at $\pm\alpha i$. Also note that unlike in the previous homework, the transfer function is proper but not *strictly proper*. We therefore begin by rewriting $H(s)$ as the sum of a constant and a strictly proper rational transfer function:

$$H(s) = 1 - \frac{4\alpha s}{s^2 + 4\alpha s + \alpha^2}$$

Next, we decompose the second term (the strictly proper part) as a partial fraction expansion:

$$\begin{aligned} \frac{4\alpha s}{s^2 + 4\alpha s + \alpha^2} &= \frac{4\alpha s}{(s + (2 + \sqrt{3})\alpha)(s + (2 - \sqrt{3})\alpha)} \\ &= \frac{a}{s + (2 + \sqrt{3})\alpha} + \frac{b}{s + (2 - \sqrt{3})\alpha} \end{aligned}$$

where $a = \frac{4 + 2\sqrt{3}}{\sqrt{3}}\alpha$ and $b = \frac{2\sqrt{3} - 4}{\sqrt{3}}\alpha$. We can now compute the impulse response of the system as:

$$h(t) = \delta(t) + ae^{-(2+\sqrt{3})\alpha t} + be^{-(2-\sqrt{3})\alpha t}$$

- Consider state variables x_1 , x_2 and x_3 defined as

$$\begin{aligned}x_1 &= V_2 - V_1 \\x_2 &= V_2 - V_3 \\x_3 &= V_4\end{aligned}$$

By denoting the input voltage V_1 by the variable u , noting that $V_3 = x_1 - x_2 + u$, substituting the expressions for V_1 , V_2 , V_3 , V_4 , and $V_2 - V_3$ in terms of the state variables and u in equations (1)-(3), and rearranging, we get:

$$\dot{x}_1 + \dot{x}_2 + 2\alpha(x_1 + u) = 0 \quad (7)$$

$$\dot{x}_2 = \alpha(x_1 - x_2 + u) - \alpha x_3 \quad (8)$$

$$2\dot{x}_3 = -2\alpha x_3 + \alpha u + \alpha(x_1 - x_2 + u) \quad (9)$$

Plugging in the expression for \dot{x}_2 from (8) back into (7) and rearranging we get the following state-space description of the system:

$$\begin{aligned}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} -3\alpha & \alpha & \alpha \\ \alpha & -\alpha & -\alpha \\ 1/2\alpha & -1/2\alpha & -\alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -3\alpha \\ \alpha \\ \alpha \end{bmatrix} u \\ y &= [1 \quad -1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + u\end{aligned}$$

Remark: Comparing the number of state variables (3) in this description to the order of the transfer function (order =2) obtained in part 2 of this exercise, we notice a discrepancy. The realization given here is in fact "non-minimal". We defer further discussion until later in the course.

Problem 2

- Recall that the zeroes of $H(s)$ are the roots of the polynomial $s + 2$ while the poles are the roots of the polynomial $(s + 3)(s^2 + 2s + 2)$. Thus the system has a real zero at $s = -2$, a real pole at $s = -3$ and a pair of complex conjugate poles at $s = -1 \pm i$.
- Since $H(s)$ is a stable transfer function (all its poles are in the open left half plane), its DC gain, $1/3$, is the steady state value of its step response.
- The Laplace transform of the output signal corresponding to input $u(t) = \sin t$ is given by:

$$\begin{aligned}Y(s) &= \frac{s + 2}{(s + 3)(s^2 + 2s + 2)(s^2 + 1)} \\ &= -\frac{1}{50} \cdot \frac{1}{s + 3} + \frac{1}{50} \cdot \frac{14s + 16}{s^2 + 2s + 2} + \frac{1}{50} \cdot \frac{9 - 13s}{s^2 + 1} \\ &= \frac{1}{50} \left(-\frac{1}{s + 3} + 14 \frac{s + 1}{(s + 1)^2 + 1} + 2 \frac{1}{(s + 1)^2 + 1} - 13 \frac{s}{s^2 + 1} + 9 \frac{1}{s^2 + 1} \right)\end{aligned}$$

Hence the output is given by:

$$\begin{aligned} y(t) &= \frac{1}{50} \left(-e^{-3t} + 14e^{-t} \cos t + 2e^{-t} \sin t - 13 \cos t + 9 \sin t \right), \quad t \geq 0 \\ &= \frac{1}{50} \left(-e^{-3t} + (14e^{-t} - 13) \cos t + (2e^{-t} + 9) \sin t \right), \quad t \geq 0 \end{aligned}$$

Problem 3

- NO. The system has memory as the instantaneous value of u at time t , $u(t)$, is not sufficient to compute $y(t)$.
- YES. Let y_1 and y_2 be the outputs of the system corresponding to inputs u_1 and u_2 , respectively. For any scalars α_1 and α_2 , the output of the system to input $\alpha_1 u_1(t) + \alpha_2 u_2(t)$ is given by:

$$\begin{aligned} y(t) &= \int_0^\infty e^{-(t-\tau)} (\alpha_1 u_1(\tau) + \alpha_2 u_2(\tau)) d\tau \\ &= \alpha_1 \int_0^\infty e^{-(t-\tau)} u_1(\tau) d\tau + \alpha_2 \int_0^\infty e^{-(t-\tau)} u_2(\tau) d\tau \\ &= \alpha_1 y_1(t) + \alpha_2 y_2(t) \end{aligned}$$

which satisfies the principle of superposition.

- YES. Let $y_o(t)$ be the response of the system to some input $u_o(t)$, and let's compute the system response \bar{y} to a shifted input $u_o(t - T)$. We have

$$\begin{aligned} \bar{y}(t) &= \int_0^\infty e^{-(t-\tau)} u_o(\tau - T) d\tau \\ &= \int_{-T}^\infty e^{-(t-\tau'-T)} u_o(\tau') d\tau' \\ &= \int_0^\infty e^{-((t-T)-\tau')} u_o(\tau') d\tau' \\ &= y_o(t - T) \end{aligned}$$

where the second equality follows by a change of variable $\tau' = \tau - T$ and the third equality assumes that *only right-sided input signals are allowed*, which is reasonable in this setting.

- NO. We have

$$\begin{aligned} y(t) &= \int_0^\infty e^{-(t-\tau)} u(\tau) d\tau \\ &= \int_0^t e^{-(t-\tau)} u(\tau) d\tau + \int_t^\infty e^{-(t-\tau)} u(\tau) d\tau \end{aligned}$$

It is clear from the second term that the value of y at time t depends on future values of the input. Hence the system is not causal.