Notes for Signals and Systems

2.2 The Class of CT Singularity Signals

The basic singularity signal is the *unit impulse*, $\delta(t)$, a signal we invent in order to have the following *sifting property* with respect to ordinary signals, $x(t)$:

$$\int_{-\infty}^{\infty} x(t)\delta(t)\, dt = x(0) \qquad (2.1)$$

That is, $\delta(t)$ causes the integral to "sift out" the value of $x(0)$. Here $x(t)$ is any continuous-time signal that is a continuous function at $t = 0$, so that the value of $x(t)$ at $t = 0$ is well defined. For example, a unit step, or the signal $x(t) = 1/t$, would not be eligible for use in the sifting property. (However, some treatments do allow a finite jump in $x(t)$ at $t = 0$, as occurs in the unit step signal, and the sifting property is defined to give the mid-point of the jump. That is,

$$\int_{-\infty}^{\infty} x(t)\delta(t)\, dt = \frac{x(0^+) + x(0^-)}{2}$$

For example, if the signal is the unit step, then the sift would yield $1/2$.)

A little thought, reviewed in detail below, shows that $\delta(t)$ cannot be a function in the ordinary sense. However, we develop further properties of the unit impulse by focusing on implications of the sifting property, while insisting that in other respects $\delta(t)$ behave in a manner consistent with the usual rules of arithmetic and calculus of ordinary functions.

- *Area*

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \qquad (2.2)$$

Considering the sifting property with the signal $x(t) = 1$, for all *t,* we see the unit impulse must satisfy (2.2).

- *Time values*

$$\delta(t) = 0, \text{ for } t \neq 0 \qquad (2.3)$$

By considering $x(t)$ to be any signal that is continuous at $t = 0$ with $x(0) = 0$, for example, the signals $x(t) = t, t^2, t^3, \ldots$, it can be shown that there is no contribution to the integral in (2.1) for nonzero values of the integration variable. This indicates that the impulse must be zero for nonzero arguments. Obviously $\delta(0)$ cannot be zero, and indeed it must have, in some sense, infinite value. That is, the unit impulse is zero everywhere except $t = 0$, and yet has unit area. This makes clear the fact that we are dealing with something outside the realm of basic calculus.

Notice also that these first two properties imply that

$$\int_{-a}^{a} \delta(t)\, dt = 1$$

for any $a > 0$.

- *Scalar multiplication*

We treat the scalar multiplication of an impulse the same as the scalar multiplication of an ordinary signal. To interpret the sifting property for $a\,\delta(t)$, where *a* is a constant, note that the properties of integration imply

$$\int_{-\infty}^{\infty} x(t)\,[a\delta(t)]\, dt = a \int_{-\infty}^{\infty} x(t)\,\delta(t)\, dt = a\, x(0)$$

The usual terminology is that $a\delta(t)$ is an "impulse of area *a*," based on choosing $x(t) = 1$, for all *t*, in the sifting expression.

- *Signal Multiplication*

$$z(t)\delta(t) = z(0)\delta(t)$$

When a unit impulse is multiplied by a signal *z*(*t*), which is assumed to be continuous at $t = 0$, the sifting property gives

$$\int_{-\infty}^{\infty} x(t)\,[z(t)\delta(t)]\, dt = \int_{-\infty}^{\infty} [x(t)z(t)]\,\delta(t)\, dt = x(0)z(0)$$

This is the same as the result obtained when the unit impulse is multiplied by the constant $z(0)$,

$$\int_{-\infty}^{\infty} x(t)[z(0)\delta(t)]\, dt = z(0) \int_{-\infty}^{\infty} x(t)\delta(t)\, dt = z(0)x(0)$$

Therefore we conclude the signal multiplication property shown above.

- *Time shift*

We treat the time shift of an impulse the same as the time shift of any other signal. To interpret the sifting property for the time shifted unit impulse, $\delta(t - t_o)$, a change of integration variable from *t* to $\tau = t - t_o$ gives

$$\int_{-\infty}^{\infty} x(t)\,\delta(t - t_o)\, dt = \int_{-\infty}^{\infty} x(\tau + t_o)\,\delta(\tau)\, d\tau = x(t_o)$$

This property, together with the function multiplication property gives the more general statement

$$z(t)\delta(t - t_o) = z(t_o)\delta(t - t_o)$$

where t_o is any real constant and $z(t)$ is any ordinary signal that is a continuous function of t at $t = t_o$.

- *Time scale*

Since an impulse is zero for all nonzero arguments, time scaling an impulse has impact only with regard to the sifting property where, for any nonzero constant a,

$$\int_{-\infty}^{\infty} x(t)\,\delta(at)\,dt = \frac{1}{|a|}x(0), \quad a \neq 0$$

To justify this expression, assume first that $a > 0$. Then the sifting property must obey, by the principle of consistency with the usual rules of integration, and in particular with the change of integration variable from t to $\tau = at$,

$$\int_{-\infty}^{\infty} x(t)\,\delta(at)\,dt = \frac{1}{a}\int_{-\infty}^{\infty} x(\tau/a)\,\delta(\tau)\,d\tau = \frac{1}{a}x(0), \quad a > 0$$

A similar calculation for $a < 0$, where now the change of integration variable yields an interchange of limits, gives

$$\int_{-\infty}^{\infty} x(t)\,\delta(at)\,dt = \frac{1}{a}\int_{\infty}^{-\infty} x(\tau/a)\,\delta(\tau)\,d\tau$$

$$= -\frac{1}{a}\int_{-\infty}^{\infty} x(\tau/a)\,\delta(\tau)\,d\tau = -\frac{1}{a}x(0), \quad a < 0$$

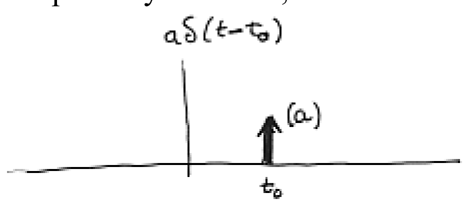
These two cases can be combined into one expression given above. Thus the sifting property leads to the definition:

$$\delta(at) = \frac{1}{|a|}\delta(t), \quad a \neq 0$$

- *Symmetry*

Note that the case $a = 1$ in time scaling gives the result that $\delta(-t)$ acts in the sifting property exactly as $\delta(t)$, so we regard the unit impulse as an "even function." Other interpretations are possible, but we will not go there.

We graphically represent an impulse by an arrow, as shown below.



(If the area of the impulse is negative, $a < 0$, sometimes the arrow is drawn pointing south.)

We could continue this investigation of properties of the impulse, for example, using the calculus consistency principle to figure out how to interpret $\delta(at - t_o)$, $z(t)\delta(at)$, and so on. But we only need the properties justified above, and two additional properties that are simply wondrous. These include an extension of the sifting property that violates the continuity condition:

- *Special Property 1*

$$\int_{-\infty}^{\infty} \delta(\tau)\delta(t-\tau)\, d\tau = \delta(t)$$

Note here that the integration variable is τ, and t is any real value. Even more remarkable is an expression that relates impulses and complex exponentials:

- *Special Property 2*

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{jt\omega}\, d\omega$$

Note here that the integral simply does not converge in the usual sense of basic calculus, since $|e^{jt\omega}| = 1$ for any (real) values of t and ω.

Remark Our general approach to these impulse properties will be "don't think about impulses… simply follow the rules." However, to provide a bit of explanation, with little rigor, we briefly discuss one of the mathematical approaches to the subject. To arrive at the unit impulse, consider the possibility of an infinite sequence of functions, $d_n(t), \quad n = 1, 2, 3, \ldots$, that have the unit-area property

$$\int_{-\infty}^{\infty} d_n(t)\, dt = 1, \quad n = 1, 2, 3, \ldots$$

and also have the property that for any other function $x(t)$ that is continuous at $t = 0$,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} d_n(t) x(t)\, dt = x(0)$$

Here the limit involves a sequence of numbers defined by ordinary integrals, and can be interpreted in the usual way. However we next interchange the order of the limit and the integration, without proper justification, and view $\delta(t)$ as "some sort" of limit:

$$\delta(t) = \lim_{n\to\infty} d_n(t)$$

This view is useful for intuition purposes, but is dangerous if pursued too far by elementary means. In particular, for the sequences of functions $d_n(t)$ typically considered, the limit does not exist in any usual sense.

Examples Consider the rectangular-pulse signals

$$d_n(t) = \begin{cases} n, & \frac{-1}{2n} < t < \frac{1}{2n} \\ 0, & else \end{cases}, \quad n = 1, 2, 3, \ldots$$

The pulses get taller and thinner as n increases, but clearly every $d_n(t)$ is unit area, and the mean-value theorem can be used to show

$$\int_{-\infty}^{\infty} d_n(t)x(t)\,dt = n\int_{-1/(2n)}^{1/(2n)} x(t)\,dt \approx n\frac{x(0)}{n}$$

with the approximation getting better as *n* increases. Thus we can casually view a unit impulse as the limit, as $n \to \infty$, of these unit-area rectangles. A similar example is to take $d_n(t)$ to be a triangle of height *n*, width 2/*n*, centered at the origin. But it turns out that a more interesting example is to use the *sinc* function defined by

$$\text{sinc}(t) = \frac{\sin(\pi t)}{(\pi t)}$$

and let

$$d_n(t) = n\sin c(nt) \quad , \quad n = 1, 2, 3, \ldots$$

It can be shown, by evaluating an integral that is not quite elementary, that these signals all have area 2π, and that the sifting property

$$\int_{-\infty}^{\infty} d_n(t)x(t)\,dt \approx x(0)$$

is a better and better approximation as *n* grows without bound. Therefore we can view an impulse of area 2π, that is, $2\pi\delta(t)$, as a limit of these functions. This sequence of *sinc* signals is displayed in the applet below for a range of *n*, and you can get a pictorial view of how an impulse might arise from *sinc's* as *n* increases, in much the same way as the impulse arises from height *n*, width *1/n ,* rectangular pulses as *n* increases.

Family of Sincs

Remark Special Property 2 can be intuitively understood in terms of our casual view of impulses as follows. Let

$$\begin{aligned} d_W(t) &= \tfrac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\,d\omega \\ &= \tfrac{1}{2\pi}\int_{-W}^{W} [\cos(\omega t) + j\sin(\omega t)]\,d\omega \\ &= \tfrac{1}{2\pi}\int_{-W}^{W} \cos(\omega t)\,d\omega + \tfrac{j}{2\pi}\int_{-W}^{W} \sin(\omega t)\,d\omega \end{aligned}$$

Using the fact that a sinusoid is an odd function of its argument,

$$\begin{aligned} d_W(t) &= \tfrac{1}{\pi}\int_{0}^{W} \cos(\omega t)\,d\omega \\ &= \tfrac{1}{\pi}\frac{\sin(Wt)}{t} \\ &= \tfrac{W}{\pi}\,\text{sinc}(\tfrac{Wt}{\pi}) \end{aligned}$$

This $d_W(t)$ can be shown to have unit area for every $W > 0$, again by a non-elementary integration, and again the sifting property is approximated when W is large. Therefore the Special Property 2 might be expected. The applet below shows a plot of $d_W(t)$ as W is varied, and provides a picture of how the impulse might arise as W increases.

Another Sinc Family

• *Additional Singularity Signals*

From the unit impulse we generate additional singularity signals using a generalized form of calculus. Integration leads to

$$\int_{-\infty}^{t} \delta(\tau)\, d\tau = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

which is the familiar *unit-step function*, $u(t)$. (We leave the value of *u(0)*, where the jump occurs, freely assignable following our general policy.)
The "running integral" in this expression actually can be interpreted graphically in very nice way. And a variable change from τ to $\sigma = t - \tau$ gives the alternate expression

$$u(t) = \int_{0}^{\infty} \delta(t-\sigma)\, d\sigma$$

Analytically this can be viewed as an application of a sifting property applied to the case $x(t) = u(t)$:

$$\int_{0}^{\infty} \delta(t-\sigma)\, d\sigma = \int_{-\infty}^{\infty} u(\sigma)\delta(t-\sigma)\, d\sigma = \int_{-\infty}^{\infty} u(t-\sigma)\delta(\sigma)\, d\sigma = u(t)$$

This is not, strictly speaking, legal for $t = 0$, because of the discontinuity there in $u(t)$, but we ignore this issue.

By considering the running integral of the unit-step function, we arrive at the *unit-ramp*:

$$\int_{-\infty}^{t} u(\tau)\, d\tau = \begin{cases} 0, & t < 0 \\ t, & t \geq 0 \end{cases}$$
$$= tu(t)$$

Often we write this as $r(t)$. Notice that the unit ramp is a continuous function of time, though it is unbounded.
Continuing,

$$\int_{-\infty}^{t} r(\tau)\, d\tau = \begin{cases} 0, & t < 0 \\ t^2/2, & t \geq 0 \end{cases}$$
$$= \frac{t^2}{2} u(t)$$

which might be called the *unit parabola*, $p(t)$. We stop here, as further integrations yield signals little used in the sequel.

We can turn matters around, using differentiation and the fundamental theorem of calculus. Clearly,

$$\frac{d}{dt}p(t)=\frac{d}{dt}\int_{-\infty}^{t} r(\tau)\,d\tau = r(t)$$

and this is a perfectly legal application of the fundamental theorem since the integrand, $r(t)$, is a continuous function of time. However, we go further, cheating a bit on the assumptions, since the unit step is not continuous, to write

$$\frac{d}{dt}r(t)=\frac{d}{dt}\int_{-\infty}^{t} u(\tau)\,d\tau = u(t)$$

That this cheating is not unreasonable follows from a plot of the unit ramp, *r(t)*, and then a plot of the slope at each value of *t*.
Cheating more, we also write

$$\frac{d}{dt}u(t)=\frac{d}{dt}\int_{-\infty}^{t} \delta(\tau)\,d\tau = \delta(t)$$

Again, a graphical interpretation makes this seem less unreasonable.

We can also consider "derivatives" of the unit impulse. The approach is again to demand consistency with other rules of calculus, and use integration by parts to interpret the "sifting property" that should be satisfied. We need go no further than the first derivative, where

$$\begin{aligned}\int_{-\infty}^{\infty} x(t)[\tfrac{d}{dt}\,\delta(t)]\,dt &= x(t)\,\delta(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \dot{x}(t)\,\delta(t)\,dt\\ &= -\dot{x}(0)\end{aligned}$$

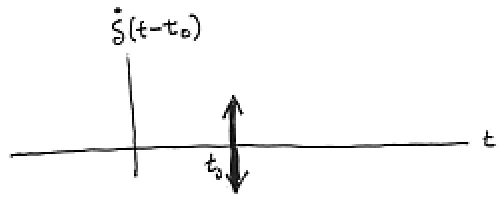
assuming of course that $\dot{x}(t)$ is continuous at $t = 0$. This unit-impulse derivative is usually called the *unit doublet*, and denoted $\dot{\delta}(t)$. Various properties can be deduced, just as for the unit impulse. For example, choosing $x(t)$ to be identically unity, the sifting property for the doublet gives

$$\int_{-\infty}^{\infty} \dot{\delta}(t)\,dt = 0$$

In other words, the doublet has zero area – a true ghost. It is also easy to verify the property

$$\int_{-\infty}^{\infty} x(t)\,\dot{\delta}(t-t_o)\,dt = -\,\dot{x}(t_o)$$

and, finally, we sketch the unit doublet as shown below.



All of the “generalized calculus” properties can be generalized in various ways. For example, the product rule gives

$$\frac{d}{dt}\left[t\,u(t)\right] = 1\,u(t) + t\,\delta(t)$$
$$= u(t)$$

where we have used the multiplication rule to conclude $t\delta(t) = 0$. As another example, the chain rule gives

$$\frac{d}{dt}u(t - t_o) = \delta(t - t_o)$$

Remark These matters can be taken too far, to a point where ambiguities begin to overwhelm and worrisome liberties must be taken. For example, using the product rule for differentiating, and ignoring the fact that $u^2(t)$ is the same signal as $u(t)$,

$$\tfrac{d}{dt}u^2(t) = \dot{u}(t)u(t) + u(t)\dot{u}(t) = 2u(t)\delta(t)$$

The multiplication rule for impulses does not apply, since $u(t)$ is not continuous at $t = 0$, and so we are stuck. However if we interpret $u(0)$ as ½, the midpoint of the jump, we get a result consistent with $\dot{u}(t) = \delta(t)$. We will not need to take matters this far, however, since we use the “generalized calculus” only for rather simple signals.