The Johns Hopkins University
Department of Electrical and Computer Engineering

520.353 — Control Systems

20 December 1994

Final Exam

Name: ____

SSN: ______________

Instructions (Read carefully!)

1. Part 1 is required of all students.
2. Attempt any five of the six questions in Part 2.
3. This exam is closed-book, closed-notes, but the use of three 8.5×11 inch, handwritten sheets is allowed.
4. Where proofs are required, give neat, step-by-step solutions that include all details and are based on first principles. Where computations are required, give neat, step-by-step solutions that include all the details. Where graphs are needed, make sure that they are neat, and clearly labelled.
5. Good Luck!

Marks

Question	Maximum	Marks
Part 1	30	
Part 2.1	14	
2.2	14	
2.3	14	
2.4	14	
2.5	14	
2.6	14	
Total	100	

Part 1 – Required of all students. $\boxed{30}$

The position of a satellite in earth orbit, with mass m can be described at any time t by the polar coordinates $r(t)$ (the radius), $\theta(t)$ (the equatorial angle), and $\phi(t)$ (the azimuthal angle) and the variables $u_r(t)$, $u_\theta(t)$ and $u_\phi(t)$ which are thrusts which may be applied in each of the three orthogonal directions by small rocket engines or gas jets.

Assuming the earth is stationary, the equations of motion of the satellite are given by:

$$\begin{aligned} m\left(\ddot{r} - r\dot{\theta}^2\cos^2\phi - r\dot{\phi}^2 + \frac{k}{r^2}\right) &= u_r && (1)\\ m\left(\ddot{\theta}r^2\cos^2\phi + 2r\dot{r}\dot{\theta}\cos^2\phi - 2r^2\dot{\theta}\dot{\phi}\cos\phi\sin\phi\right) &= (r\cos\phi)\,u_\theta && (2)\\ m\left(\ddot{\phi}r^2 + r^2\dot{\theta}^2\cos\phi\sin\phi + 2r\dot{r}\dot{\phi}\right) &= ru_\phi && (3) \end{aligned}$$

For communications satellites, a particularly useful set of solutions is that giving a *circular, equatorial orbit*; that is:

$$r = r_0, \qquad \dot{r} = 0, \qquad \theta(t) = \omega t, \qquad \dot{\theta} = \omega, \qquad \phi(t) = 0, \qquad \dot{\phi}(t) = 0$$

where the radius r_0 and the angular velocity ω are related by

$$r_0^3\omega^2 = k$$

and all the thrusts are zero.

(a) Show that this is, in fact, a solution of equations (1)–(3) above. $\boxed{4}$

(b) Linearize equation (3):

$$m\left(\ddot{\phi}r^2 + r^2\dot{\theta}^2\cos\phi\sin\phi + 2r\dot{r}\dot{\phi}\right) = ru_\phi$$

In doing this show that, in the linear approximation, the azimuthal angle does not depend on either θ or r. Using the state vector $x := \begin{bmatrix}\phi \\ \dot{\phi}\end{bmatrix}$, express this as a standard state-space system with input $u_\phi(t)$ and output $\phi(t)$. $\boxed{8}$

(c) What kind of stability (if any) does the matrix $A = \begin{bmatrix}0 & 1 \\ -\omega^2 & 0\end{bmatrix}$ exhibit? $\boxed{4}$

(d) For the linear system in part (c), show that $\boxed{4}$

$$e^{At} = \begin{bmatrix} \cos\omega t & \frac{-1}{\omega}\sin\omega t \\ \omega\sin\omega t & \cos\omega t \end{bmatrix}$$

(e) Assume zero thrust ($u_\phi(t) = 0$). Show that any perturbation (i.e. $x_0 \neq 0$) results in sinusoidal oscillations about the zero value. $\boxed{4}$

(f) Is the system controllable? Is it observable? Note, if you were unable to solve part (b) for the matrices B and C, assume that $B = \left[\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right]$ and $C = \left[\begin{smallmatrix} c_1 & c_2 \end{smallmatrix}\right]$. $\boxed{6}$

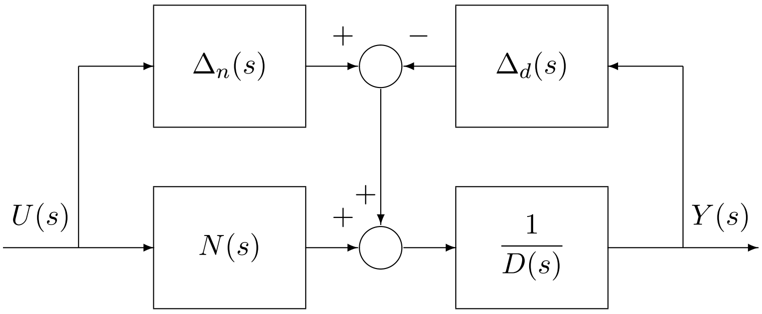
2.1 – **Do 5 out of the 6 problems in Part 2** $\boxed{14}$

Using Laplace Transforms, solve for the function $f(t)$, $t \geq 0$ which satisfies the following integral equation:

$$f(t) = t^3 + \int_0^t \sin(t-\tau) f(\tau)\, d\tau$$

2.2 – **Do 5 out of the 6 problems in Part 2** $\boxed{14}$

Write down the transfer function between the external input u and the external output y for the system depicted below. Note that N, D, Δ_n and Δ_d are all polynomials in s.



2.3 – **Do 5 out of the 6 problems in Part 2** $\boxed{14}$

Find the departure and arrival angles for the poles and zeros in the root locus diagram for the system

$$(PCF)(s) = \frac{s^3}{(s^2+1)^4}$$

Note: you do NOT have to draw the diagram.

2.4 – **Do 5 out of the 6 problems in Part 2** $\boxed{14}$

In each of the following four plots, draw a Nyquist plot of a system satisfying the desired properties:

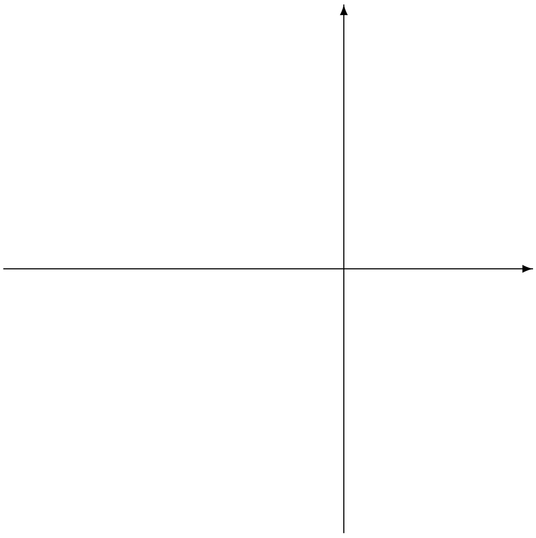
(a) $PCF(s)$ is stable; the closed-loop system has 20dB gain margin and 60^o phase margin (assume that $k = 1$.) $\boxed{3}$

(b) $k(PCF)(s)$ has one pole in $Re(s) > 0$. The closed-loop system is stable for values of k between 5 and 10. $\boxed{3}$

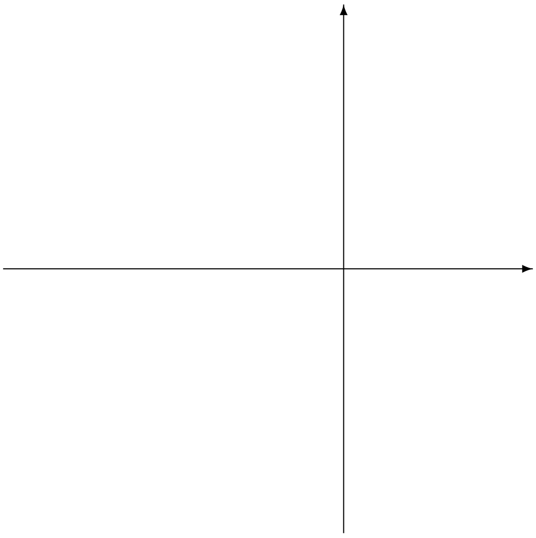
(c) $PCF(s)$ is stable; the closed-loop system has infinite gain margin, but very small phase margin. $\boxed{3}$

(d) $PCF(s)$ is stable. The closed-loop system is stable for $k = 1$. When the gain is reduced by 20%, the system goes unstable. After a further reduction of 30% of the original value (that is, to $k = 0.5$), the system again becomes stable. No further regions of stability are observed as k is varied. $\boxed{5}$

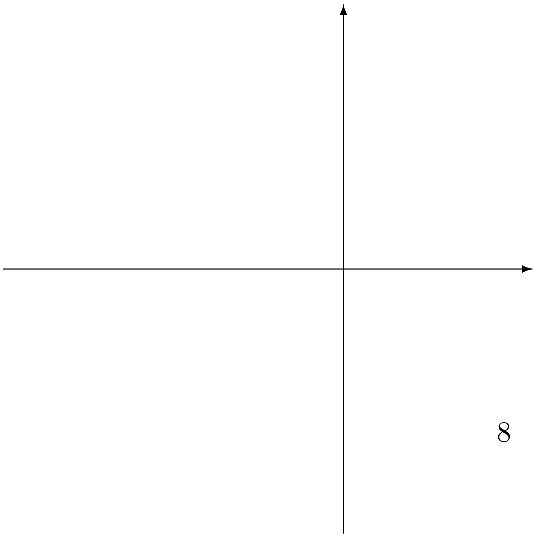
a)



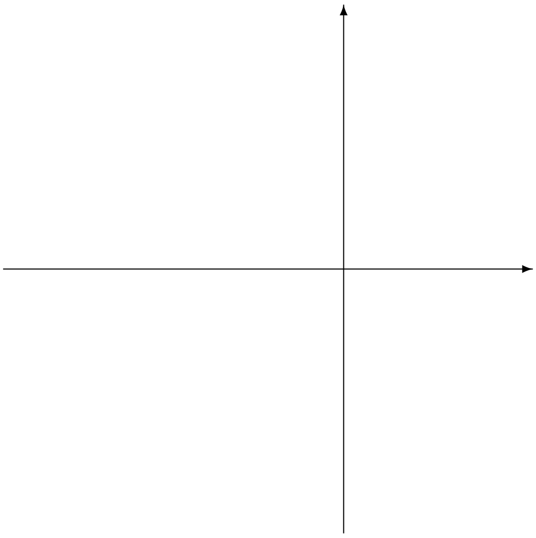
b)



c)



d)



2.5 – Do 5 out of the 6 problems in Part 2 $\boxed{14}$

Find e^{At} for the matrix

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Hint: This can be solved with very little computation.

2.6 – **Do 5 out of the 6 problems in Part 2** $\boxed{14}$

Give an example of a fourth order system (A, B, C) that satisfies *all* of these conditions: a) The system is controllable; b) the system is not observable; c) the matrix A is stable; d) the matrix A is not asymptotically stable; e) the transfer function $G(s) = C(sI - A)^{-1}B$ is UBIBOS and of second order.

Hint: Consider systems already in modal form.