The Johns Hopkins University
Department of Electrical and Computer Engineering
520.353 — Control Systems
Fall 2007

Problem Set Number 8

Due: Wednesday, December 5, 2007 in class.

Problems:

1. Determine the stability properties (i.e. are the matrices stable and/or asymptotically stable) of:

(a) $A_1 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$, (b) $A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$,

(c) $A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, (d) $A_4 = \begin{bmatrix} -1 & -2 \\ 2 & 2 \end{bmatrix}$,

(e) $A_5 = \begin{bmatrix} 1 & 2 \\ -2 & -2 \end{bmatrix}$, (f) $A_6 = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$.

Solution.

(a) The eigenvalues are at 0 (simple) and -1. Hence the matrix is stable (S) but not asymptotically stable (AS).

(b) The eigenvalues are both at 0. The Jordan block for this eigenvalue is of size 2. Hence, the matrix is not S or AS.

(c) The eigenvalues are both at 0. There are two Jordan blocks for this eigenvalue both of size 1. Hence, the matrix is S but not AS.

(d) The characteristic polynomial for this matrix is:

$$(\lambda + 1)(\lambda - 2) + 4 = \lambda^2 - \lambda + 2$$

Because the coefficient for the λ term is negative, the system is not S or AS.

(e) The characteristic polynomial for this matrix is:

$$(\lambda - 1)(\lambda + 2) + 4 = \lambda^2 + \lambda + 2$$

Because all coefficients are positive, the system is both S and AS.

(f) Both eigenvalues are at -2, hence the system is S and AS.

2. The linearized equations of motion for a satellite are

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

The inputs u_1 and u_2 are the radial and tangential thrusts, the state variables x_1 and x_3 are the radial and angular deviations from the reference (circular) orbit, and the outputs y_1 and y_3 are the radial and angular measurements, respectively.

(a) Show that the systems is controllable using only a single input. Which one is it?

(b) Show that the system is observable using only one measurement. Which one?

Solution.

(a) Let's first determine whether it is controllable from u_1:

$$\begin{aligned}\mathcal{C}_1 &= \begin{bmatrix} B_1 & AB_1 & A^2B_1 & A^3B_1 \end{bmatrix}\\ &= \begin{bmatrix} \begin{bmatrix} 0\\1\\0\\0 \end{bmatrix} & \begin{bmatrix} 1\\0\\0\\-2\omega \end{bmatrix} & \begin{bmatrix} 0\\-w^2\\-2\omega\\0 \end{bmatrix} & \begin{bmatrix} -w^2\\0\\0\\+2\omega^3 \end{bmatrix} \end{bmatrix}\end{aligned}$$

Note that the second and fourth columns are not linearly independent, which means that the matrix is not invertible and the system is not controllable from u_1. In contrast, with u_2:

$$\begin{aligned}\mathcal{C}_2 &= \begin{bmatrix} B_2 & AB_2 & A^2B_2 & A^3B_2 \end{bmatrix}\\ &= \begin{bmatrix} \begin{bmatrix} 0\\0\\0\\1 \end{bmatrix} & \begin{bmatrix} 0\\2\omega\\1\\0 \end{bmatrix} & \begin{bmatrix} 2\omega\\0\\0\\-4\omega^2 \end{bmatrix} & \begin{bmatrix} 0\\-2w^3\\-4\omega^2\\0 \end{bmatrix} \end{bmatrix}\end{aligned}$$

and

$$\det \mathcal{C}_2 = -12w^4 \neq 0$$

which implies that the system is controllable from u_2.

(b) Proceeding as above with the C matrix:

$$\mathcal{O}_1 = \begin{bmatrix} C_1\\ C_1A\\ C_1A^2\\ C_1A^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 3\omega^2 & 0 & 0 & 2\omega\\ 0 & -\omega^2 & 0 & 0 \end{bmatrix}$$

which is clearly not invertible (the third column is zero!). In contrast

$$\mathcal{O}_2 = \begin{bmatrix} C_2\\ C_2A\\ C_2A^2\\ C_2A^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & -2\omega & 0 & 0\\ -6\omega^3 & 0 & 0 & -4\omega^2 \end{bmatrix}$$

is invertible, as

$$\det \mathcal{O}_2 = 12w^4 \neq 0$$

Thus, the system is observable from the second measurement, but not the first.

3. Consider the differential equation

$$\ddot{y}(t) - p\dot{y}(t) = \dot{r}(t) - zr(t)$$

with zero initial conditions. (NB. Here, p and z are just constants.)

(a) Using

$$x_1(t) := y(t), \qquad x_2(t) := \dot{y}(t) - r(t)$$

write the differential equation in standard state-space form.

(b) Investigate the controllability and observability properties of the system you obtained in (a) as a function of the parameters z and p.

(c) Find the transfer function $G(s) := Y(s)/R(s)$ and relate the controllability/observability properties you found in (b) to $G(s)$. Specifically, when do yo lose observability and/or controllability?

Solution.

(a) The differential equation is

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \dot{y} \\ \ddot{y} - \dot{r} \end{bmatrix} = \begin{bmatrix} x_2 + r \\ p\dot{y} + \dot{r} - zr - \dot{r} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & p \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ (p-z) \end{bmatrix} r$$

Thus

$$A = \begin{bmatrix} 0 & 1 \\ 0 & p \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ (p-z) \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \text{ and } D = 0.$$

(b)

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & p-z \\ p-z & p(p-z) \end{bmatrix}$$

and

$$\det \mathcal{C} = z(p-z)$$

Thus, the system is controllable iff $p \neq z$ and $z \neq 0$. Similarly

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = I$$

which implies that the system is always observable.

(c) The transfer function (using Laplace transforms) is

$$G(s) = \frac{Y(s)}{R(s)} = \frac{s-z}{s(s-p)}$$

We lose controllability in one of two ways. Either $z = 0$, in which case the transfer function is

$$\frac{s}{s(s-p)} = \frac{1}{s-p};$$

or $p = z$, in which case the transfer function is

$$\frac{s}{s(s-p)} = \frac{1}{s};$$

Both cases involve a pole-zero cancelation. This is always the case. **Controllability or observability are lost because of a pole-zero cancelation.**

4. Give an example of a fourth order system (A, B, C) that satisfies all of these conditions:

   (a) The system is controllable;

   (b) the system is not observable;

   (c) the matrix A is stable;

   (d) the matrix A is not asymptotically stable;

   (e) the transfer function
   $$G(s) = C(sI - A)^{-1}B$$
   is UBIBOS and of second order.

   Hint: Consider systems already in modal form (i.e. with a diagonal A matrix).

Solution. Write down:

$$A = \begin{bmatrix} p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & 0 \\ 0 & 0 & 0 & p_4 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}, \quad C = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix}$$

(a) The controllability matrix is

$$\mathcal{C} = \begin{bmatrix} b_1 & p_1 b_1 & p_1^2 b_1 & p_1^3 b_1 \\ b_2 & p_2 b_2 & p_2^2 b_2 & p_2^3 b_2 \\ b_3 & p_3 b_3 & p_3^2 b_3 & p_3^3 b_3 \\ b_4 & p_4 b_4 & p_4^2 b_4 & p_4^3 b_4 \end{bmatrix}$$

One way (but by no means the only one!) to make this full rank is to set $b_i = 1$, $i \in \{1, \ldots, 4\}$ and $p_i \neq p_j$ for $i, j \in \{1, \ldots, 4\}$, $i \neq j$ which leads to

$$\mathcal{C} = \begin{bmatrix} 1 & p_1 & p_1^2 & p_1^3 \\ 1 & p_2 & p_2^2 & p_2^3 \\ 1 & p_3 & p_3^2 & p_3^3 \\ 1 & p_4 & p_4^2 & p_4^3 \end{bmatrix}$$

This is a Vandermonde matrix whose determinant is

$$\det \mathcal{C} = (p_1 - p_2)(p_1 - p_3)(p_1 - p_4)(p_2 - p_4)(p_3 - p_4)$$

(b) To make the system not observable, we just set some of the c_i equal to zero, as this makes a complete column of $\mathcal{O}$ zero:

$$\mathcal{O} = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \\ c_1 p_1 & c_2 p_2 & c_3 p_3 & c_4 p_4 \\ c_1 p_1^2 & c_2 p_2^2 & c_3 p_3^2 & c_4 p_4^2 \\ c_1 p_1^3 & c_2 p_3^2 & c_3 p_3^3 & c_4 p_4^3 \end{bmatrix}$$

(c) To make the matrix stable, make $p_i \leq 0$.

(d) To make the system not AS, make one $p_i \geq 0$. To match the previous condition, I pick $p_4 = 0$, but set the others negative.

(e) To make the transfer function be of second order, I will set two of the c_i equal to zero. Moreover, to make sure that the transfer function is UBIBOS, any eigenvalues at zero should not appear. Hence, set $c_1 = c_2 = 1$, $c_3 = c_4 = 0$, $p_1 = -1$, $p_2 = -2$, $p_3 = -3$, $p_4 = 0$ will satisfy all the requirements. The transfer function is

$$\frac{c_1 b_1}{s - p_1} + \frac{c_2 b_2}{s - p_2} = \frac{2s + 3}{(s+1)(s+2)}.$$