The Johns Hopkins University
Department of Electrical and Computer Engineering
520.353 — Control Systems
Fall 2007

Problem Set Number 5

Due: Wednesday, October 31, 2007 in class.

Problems:

1. For the system:

$$k(PCF)(s) = k\frac{s^2+1}{s^3}.$$

   (a) Find the departure/arrival angles at the poles/zeros.

   (b) Plot a clean, clearly labelled root locus diagram for this system.

Solution. Recall that: $\sum \angle(s-z_i) - \sum \angle(s-p_i) = 180° + 360k°$. There are three poles at zero and zeros at $\pm j$. Let θ be the departure angles (there's three!) at the pole. We need to compute the angle between these poles ($s \approx 0$) and the two zeros. This is

$$\begin{aligned}180° + 360k° &= \sum \angle(s-z_i) - \sum \angle(s-p_i) \\ &\approx \sum \angle(0-z_i) - \sum \theta \\ &= \angle(+j) + \angle(-j) - 3\theta \\ &= 90° - 90° - 3\theta\end{aligned}$$

Solving for θ yields $\theta = 60° + 120k° = \{60°, 180°, 300°\}$.

Now we compute the arrival angle at the $+j$ zero. Let this be ϕ. Then

$$\begin{aligned}180° + 360k° &= \sum \angle(s-z_i) - \sum \angle(s-p_i) \\ &\approx \phi + \angle(j - [-j]) - 3\sum \angle(j-0) \\ &= \phi + 90° - 3 \times 90° \\ &= \phi - 180°.\end{aligned}$$

Solving for ϕ yields $\phi = 360° + 360k° = 0°$.

In the root locus diagram, all the real axis left of zero gets covered by the root locus. This yields the diagram of Figure 1.

2. Sketch the root locus with respect to β for the standard feedback system with:

$$P(s) = \frac{4}{10s+\beta}, \qquad C(s) = 5, \qquad F(s) = \frac{5}{s+7}.$$

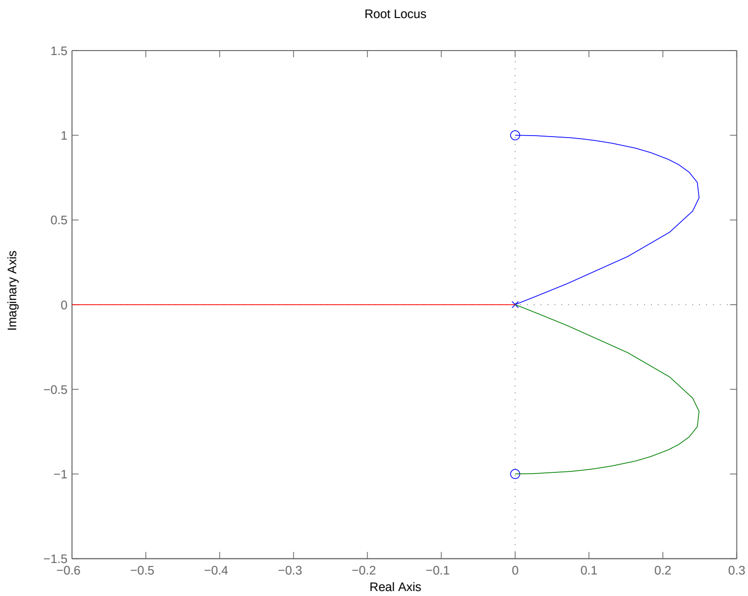

Figure 1: Root locus diagram for system of Problem 1.

Solution. We first compute the characteristic polynomial:

$$4 \times 5 \times 5 + (10s + \beta)(s + 7) = [10s(s + 7) + 100] + \beta(s + 7) = \underbrace{10(s^2 + 7s + 10)}_{D(s)} + \beta \underbrace{(s + 7)}_{N(s)}.$$

So, we draw the root locus for a system with poles at $\{-5, -2\}$ and zero at $\{-7\}$. This is pretty straightforward. The region between -2 and -5 is covered by branches that come from left and right. They meet and form a circle-like shape. They retouch the real axis to the left of the zero. There, they divide. One branch goes to the right ending at the zero at -7. The other branch goes off to $-\infty$ along the real axis. Take a look at the plot in Figure 2.

3. (a) Using all the rules discussed in class, draw the root locus for the system:

$$k(PCF)(s) = k\frac{(s + 3)}{s(s^2 + 2s + 26)}.$$

   (b) Find the gain k and the natural frequency ω_n when the system just becomes unstable.

   (c) Using Matlab and the `rlocus` command, obtain a computer printout of the root locus.

Solution. The system has poles at $\{0, -1 \pm 5j\}$ and a zero at -3.

Two branches will go to infinity at $\pm 90°$. The centroid is at:

$$\alpha = \frac{\sum p_i - \sum z_i}{2} = \frac{[(-1 - 5j) + (-1 + 5j)] - [-3]}{2} = \frac{1}{2}.$$

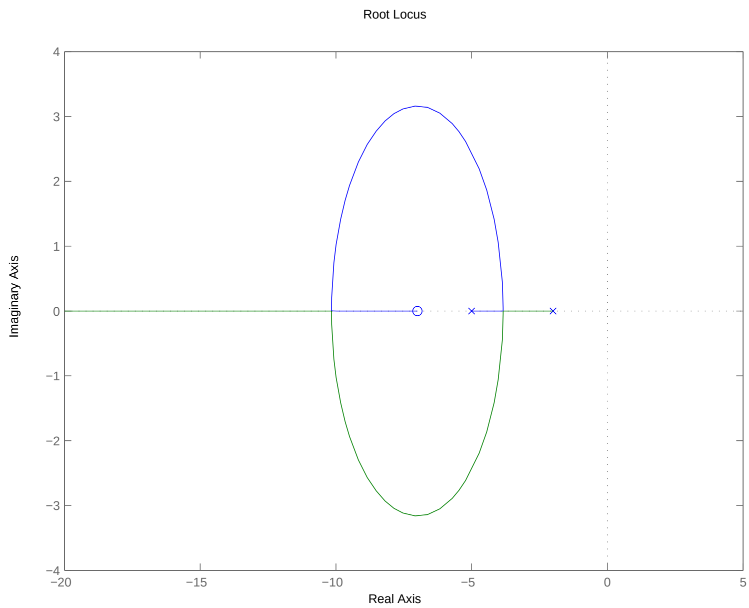

Figure 2: Root locus diagram for system of Problem 2.

Denote by θ the departure angle for the pole starting at $-1+5j$; then

$$\begin{aligned}
180^\circ &= \sum \angle(s-z_i) - \sum \angle(s-p_i) \\
&\approx \angle([-1+5j]-[-3]) - \theta - \angle([-1+5j]-[-1-5j]) - \angle([-1+5j]-0) \\
&= \angle(2+5j) - \theta - \angle(10j) - \angle(-1+5j) \\
&= \tan^{-1}(5/2) - \theta - 90^\circ - \big(90^\circ + \tan^{-1}(5)\big) \\
&\approx 68^\circ - \theta - 90^\circ - (180^\circ - 79^\circ)
\end{aligned}$$

Solving for θ yields $\theta = -303^\circ = +57^\circ$.

We need now to find the frequency and gain at which the branches hit the imaginary axis. This is easiest using the characteristic equation. When it hits the imaginary axis, there will be one real root (say at $-r$) and imaginary roots at $\pm j\omega_n$. Thus:

$$s^3 + 2s^2 + [k+26]s + 3k = (s+r)(s^2+\omega_n^2) = s^3 + rs^2 + \omega_n^2 s + r\omega_n^2.$$

Comparing term by term, we see that

$$\begin{aligned}
r &= 2 \\
k + 26 &= \omega_n^2 \\
3k &= 2\omega_n^2
\end{aligned}$$

Thus,

$$3k = 2k + 52 \Longrightarrow k = 52 \Longrightarrow \omega_n^2 = 78 \Rightarrow \omega_n \approx 8.83.$$

4. In the feedback system, take

$$P(s) = \frac{s+1}{s(s+2)}, \qquad C(s) = k, \qquad F(s) = 1.$$

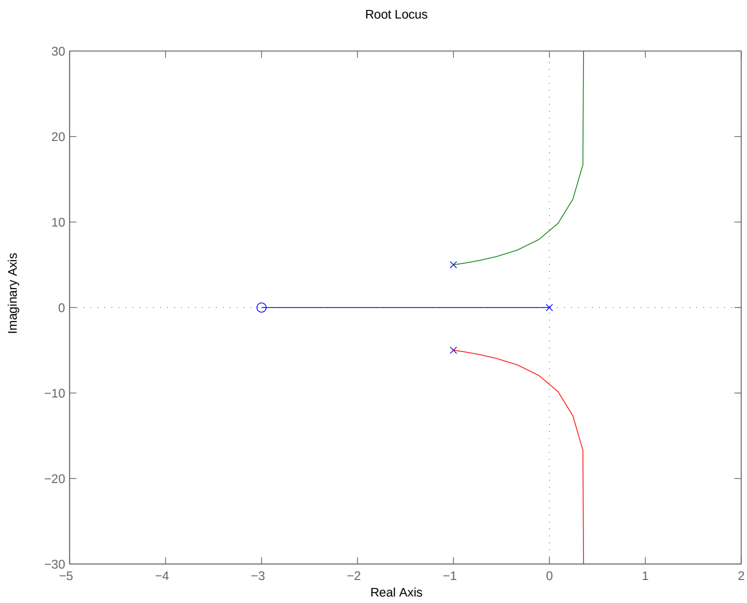

Figure 3: Root locus diagram for system of Problem 3. The angle of departure for the pole at $-1+5j$ is approximately $57°$. It appears smaller, but that is because the imaginary axis scale is quite different from that of the real axis. The angle for the pole at $-1-5j$ is $-57°$ from symmetry.

Sketch the Nyquist plot of P and determine the range of k for which the feedback system is stable.

Solution. First let's write $(PCF)(j\omega)$ in real and imaginary components:

$$\begin{aligned}(PCF)(j\omega) &= \frac{1+j\omega}{j\omega(2+j\omega)}\\ &= \frac{-j(1+j\omega)}{\omega(2+j\omega)} \times \frac{2-j\omega}{2-j\omega}\\ &= \frac{-j(2-j\omega+2j\omega+\omega^2)}{\omega(4+\omega^2)}\\ &= \frac{-j([2+\omega^2]+j\omega)}{\omega(4+\omega^2)}\\ &= \frac{\omega-j[2+\omega^2]}{\omega(4+\omega^2)}\end{aligned}$$

Thus:

$$\mathrm{Re}(PCF)(j\omega) = \frac{1}{4+\omega^2}, \qquad \mathrm{Im}(PCF)(j\omega) = -\frac{2+\omega^2}{\omega(4+\omega^2)}.$$

Now consider the Nyquist contour. Because there is a pole at zero, we need to indent around it. I will form a table:

Label	ω	$\text{Re}(PCF)(j\omega)$	$\text{Im}(PCF)(j\omega)$
A	0^+	0.25	$-\infty$
B	1	0.2	-0.6
C	$+\infty$	0^+	0^-
C′	$-\infty$	0^+	0^+
B′	-1	0.2	0.6
A′	0^-	0.25	$+\infty$

The Nyquist plot is given in Figure 4.

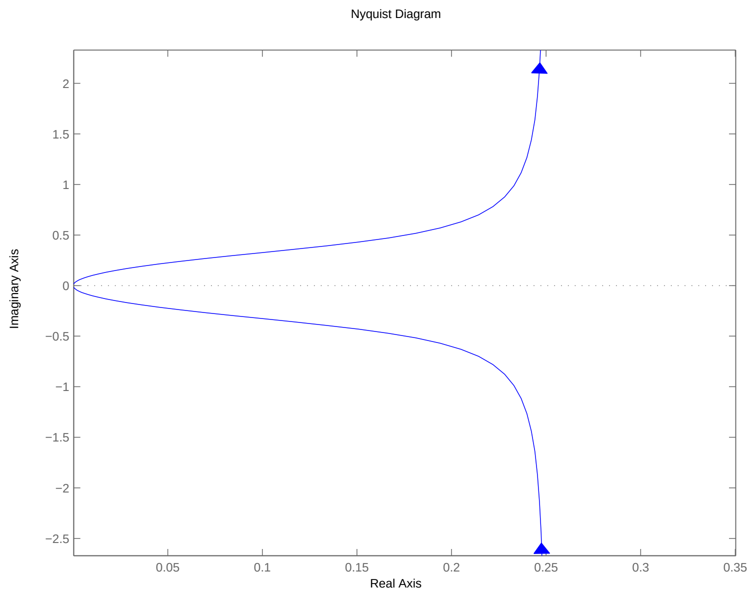

Figure 4: Nyquist plot for system of Problem 4.

To join A′ to A, we do a 180° turn CW, so that it goes from the top, clockwise, to the bottom. This means that there are two regions:

Region I: $-1/k < 0$, which has no encirclements.

Region II: $-1/k > 0$, which has one CW encirclement.

We need 0 CCW encirclement, so k must be in region I:

$$k > 0.$$

We can check this using the characteristic equation:

$$\Delta(s) = s^2 + [k+2]s + k.$$

From the s^0 term, clearly $k > 0$ is necessary and sufficient.

5. In the feedback system, take

$$P(s) = \frac{(s+2)}{s(s+1)(s-1)}, \qquad C(s) = k, \qquad F(s) = 1.$$

Sketch the Nyquist plot of P and determine the range of k, if any, for which the feedback system is stable.

Solution. We proceed as before. We compute $(PCF)(j\omega)$:

$$(PCF)(j\omega) = \frac{2 + j\omega}{j\omega(-w^2 - 1)}$$
$$= \frac{-\omega + 2j}{\omega(w^2 + 1)}$$

Thus:

$$\text{Re}(PCF)(j\omega) = -\frac{1}{1+\omega^2}, \qquad \text{Im}(PCF)(j\omega) = \frac{2}{\omega(1+\omega^2)}.$$

As above, the Nyquist contour indents around the pole at zero.

Label	ω	$\text{Re}(PCF)(j\omega)$	$\text{Im}(PCF)(j\omega)$
A	0^+	-1	$+\infty$
B	1	-0.5	1
C	$+\infty$	0^-	0^+
C′	$-\infty$	0^-	0^-
B′	-1	-0.5	-1
A′	0^-	-1	$-\infty$

with Nyquist plot given in Figure 5.

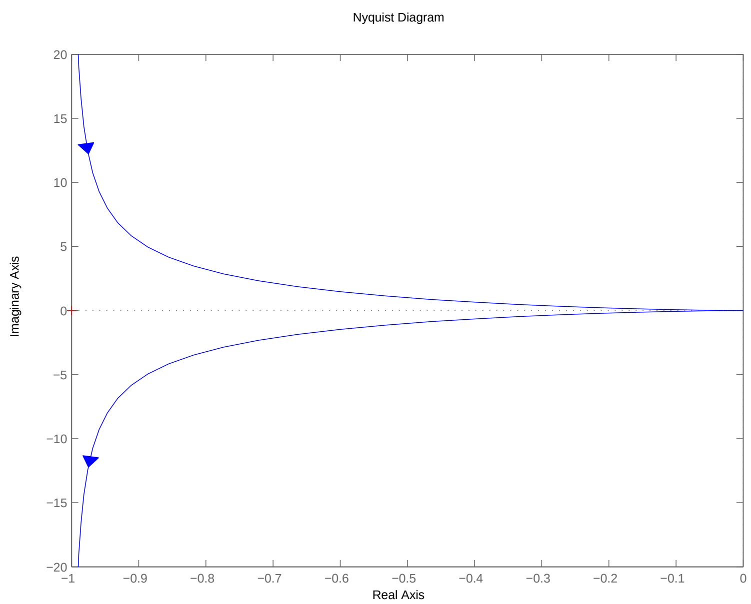

Figure 5: Nyquist plot for system of Problem 5.

To join A′ to A, we do a 180° turn CW, so that it goes from the bottom, clockwise, to the top, enclosing the region in the left half-pane. This means that there are two regions:

Region I: $-1/k < 0$, which has 1 CW encirclement.

Region II: $-1/k > 0$, which has no encirclements.

We need 1 CCW encirclement, because the pole at -1, so no k will do.

$$k > 0.$$

We can check this using the characteristic equation:

$$\Delta(s) = s^3 + [k - 1]s + 2k.$$

From the s^2 term, clearly no k will make this positive, and so the system is always unstable.