The Johns Hopkins University
Department of Electrical and Computer Engineering
520.353 — Control Systems
Fall 2007

Problem Set Number 4 — Solutions

Due: Wednesday, October 10, 2007 in class.

Reminder: Your first term test will be on October 17, 2007.

Note: Class on October 3 will be in the ECE lab (Barton 123) and will present an introduction to Matlab and Simulink for use in Control Systems

Problems:

1. Use the Routh-Hurwitz criterion to determine which values of k make the polynomial

$$s^3 + (4+k)s^2 + 6s + 12$$

have all its roots in the open left-hand plane.

Solution. Forming the Routh table:

$$\begin{array}{c|cc} s^3 & 1 & 6 \\ s^2 & 4+k & 12 \\ s^1 & b_1 & \\ s^0 & c_0 & \end{array}$$

where

$$b_1 = -\frac{\det\begin{bmatrix} 1 & 6 \\ 4+k & 12 \end{bmatrix}}{4+k} = -\frac{12 - 24 - 6k}{4+k} = \frac{6(2+k)}{4+k}$$

and

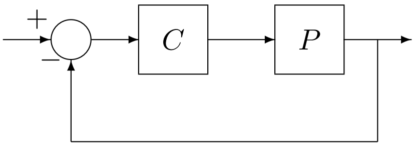
$$c_0 = -\frac{\det\begin{bmatrix} 4+k & 12 \\ b_1 & 0 \end{bmatrix}}{b_1} = 12.$$

For stability, we need the first column elements to be positive:

$$k > -4, \qquad k > -2.$$

Overall, we need $k > -2$.

2. Consider the following closed-loop system, where $P(s) = k/s$.



Does there exist a proper controller $C(s)$ such that the feedback system is stable for both values of $k = 1$ and $k = -1$? Justify your answer.

Solution. The answer is no. To show this, we look at the characteristic polynomial. Suppose that

$$C(s) = \frac{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}.$$

Note that some of the numerator's terms may be zero. Together with $P(s)$, this leads to the following characteristic polynomial:

$$\begin{aligned}\Delta(s) &= s(s^n + a_{n-1}s^{n-1} + \cdots + a_0) + k(b_n s^n + b_{n-1}s^{n-1} + \cdots + b_0) \\ &= s^{n+1} + [a_{n-1} + kb_n]s^n + \cdots + [a_0 + kb_1]s + kb_0\end{aligned}$$

Clearly, when $k = \pm 1$, the last term will be both positive and negative. But no b_0 will make this positive for both $k = \pm 1$ and so the characteristic polynomial can't be stable for both values of k.

3. For the standard feedback system with:

$$P(s) = \frac{10}{s^2 + s + 10}, \quad C(s) = \frac{K}{s}, \quad F(s) = \frac{1}{s+2};$$

find the range of positive, constant K (if it exists) for which the system is: (a) internally stable; (b) all the roots of the characteristic polynomial are in Re $s < -1$.

Solution. Let's compute the characteristic polynomial:

$$\Delta(s) = s(s+2)(s^2 + s + 10) + 10K = s^4 + 3s^3 + 12s^2 + 20s + 10K.$$

(a) Create the Routh table:

s^4	1	12	$10K$
s^3	3	20	
s^2	b_2	b_0	
s^1	c_1		
s^0	d_0		

where

$$\begin{aligned}
b_2 &= -\frac{\det\begin{bmatrix}1 & 12\\ 3 & 20\end{bmatrix}}{3} = -\frac{20 - 36}{3} = \frac{16}{3} \\
b_0 &= -\frac{\det\begin{bmatrix}1 & 10K\\ 3 & 0\end{bmatrix}}{3} = 10K \\
c_1 &= -\frac{\det\begin{bmatrix}3 & 20\\ b_2 & b_0\end{bmatrix}}{b_2} = -\frac{30K - 320/3}{16/3} = \frac{160 - 45K}{8} \\
d_0 &= -\frac{\det\begin{bmatrix}b_2 & b_0\\ c_1 & 0\end{bmatrix}}{c_1} = b_0 = 10K
\end{aligned}$$

We need:

$$160 - 45K > 0 \qquad \text{and} \qquad K > 0$$

Together, this leads to the requirement:

$$0 < K < \frac{32}{9}.$$

(b) To place the poles to the left of Re $s < -1$, we let $s' = s + 1 \iff s = s' - 1$. We then apply the Routh-Hurwitz result to:

$$\begin{aligned}
\Delta'(s') &= \Delta(s' - 1), \qquad \text{(I'm going to drop the } ' \text{ from now on)} \\
&= (s-1)^4 + 3(s-1)^3 + 12(s-1)^2 + 20(s-1) + 10K \\
&= (s^4 - 4s^3 + 6s^2 - 4s + 1) + 3(s^3 - 3s^2 + 3s - 1) + 12(s^2 - 2s + 1) \\
&\quad\; + 20(s-1) + 10K \\
&= s^4 + [-4+3]s^3 + [6 - 9 + 12]s^2 + [-4 + 9 - 24 + 20]s + [1 - 3 + 12 - 20 + 10K] \\
&= s^4 - s^3 + 9s^2 + s + [10K - 10]
\end{aligned}$$

Note how there is one term with a -1 coefficient, and so this system can never be stable.

4. Do problem 3.40 in your text.

Solution. We are suppose to find the range of K for which the roots of the polynomial:

$$s^5 + 5s^4 + 10s^3 + 10s^2 + 5s + K$$

are in the open left-hand plane. The Routh table is

s^5	1	10	5
s^4	5	10	K
s^3	b_3	b_1	
s^2	c_2	c_0	
s^1	d_1		
s^0	e_0		

where

$$b_3 = -\frac{\det \begin{bmatrix} 1 & 10 \\ 5 & 10 \end{bmatrix}}{5} = 8$$

$$b_1 = -\frac{\det \begin{bmatrix} 1 & 5 \\ 5 & K \end{bmatrix}}{5} = \frac{25 - K}{5}$$

$$c_2 = -\frac{\det \begin{bmatrix} 5 & 10 \\ b_3 & b_1 \end{bmatrix}}{b_3} = \frac{10b_3 - 5b_1}{b_3} = \frac{K + 55}{8}$$

$$c_0 = -\frac{\det \begin{bmatrix} 5 & K \\ b_3 & 0 \end{bmatrix}}{b_3} = K$$

$$d_1 = -\frac{\det \begin{bmatrix} b_3 & b_1 \\ c_2 & c_0 \end{bmatrix}}{c_2} = \frac{b_1 c_2 - b_3 c_0}{c_2} = \frac{(25 - K)(K + 55)/40 - 8K}{c_2} = \frac{-(K^2 + 350K - 1375)}{40c_2}$$

$$e_0 = -\frac{\det \begin{bmatrix} c_2 & c_0 \\ d_1 & 0 \end{bmatrix}}{d_1} = c_0 = K$$

For stability we need:

$$\begin{aligned} b_3 = 8 &> 0, && \text{always true} \\ c_2 = \frac{K+55}{8} &> 0, && \iff K > -55 \\ d_1 = \frac{-(K^2+350K-1375)}{40c_2} &> 0, && \iff -353.8854 < K < 3.8854, (\text{when } c_2 > 0) \\ e_0 = K &> 0, && \iff K > 0 \end{aligned}$$

Altogether, we need:

$$0 < K < 3.8854$$

We check this in Matlab:

```
 >> K=3; roots([1 5 10 10 5 K])

ans =
-2.1487
-1.3550 + 1.0925i
-1.3550 - 1.0925i
-0.0707 + 0.6752i
-0.0707 - 0.6752i
>> K=3.8; roots([1 5 10 10 5 K])
ans =
-2.2287
-1.3797 + 1.1685i
-1.3797 - 1.1685i
-0.0060 + 0.7222i
-0.0060 - 0.7222i
>> K=4; roots([1 5 10 10 5 K])
ans =
-2.2457
-1.3850 + 1.1848i
-1.3850 - 1.1848i
0.0078 + 0.7322i
0.0078 - 0.7322i
```

5. For the transfer function

$$G(s) = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

where $\omega_n = \sqrt{\omega_d^2 + \sigma^2}$, show that the step-response is

$$\begin{aligned} y(t) &= 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right) \\ &= 1 - e^{-\sigma t}\sqrt{1 + \frac{\sigma^2}{\omega_d^2}}\cos(\omega_d t - \beta) \end{aligned}$$

where

$$\beta = \tan^{-1}\frac{\sigma}{w_d}.$$

Solution. The output is

$$Y(s) = \frac{G(s)}{s} = \frac{\omega_n^2}{s[(s+\sigma)^2 + \omega_d^2]}$$

We do a partial fraction expansion:

$$\begin{aligned}\frac{G(s)}{s} &= \frac{\omega_n^2}{s[(s+\sigma)^2 + \omega_d^2]} \\ &= \frac{A}{s} + \frac{B(s+\sigma)}{(s+\sigma)^2+\omega_d^2} + \frac{C}{(s+\sigma)^2+\omega_d^2}\end{aligned}$$

Using residue:

$$A = \lim_{s\to 0} s\frac{G(s)}{s} = G(0) = 1.$$

To get the other terms, expand:

$$\begin{aligned}\frac{G(s)}{s} &= \frac{[(s+\sigma)^2+\omega_d^2] + Bs(s+\sigma) + Cs}{s[(s+\sigma)^2+\omega_d^2]} \\ &= \frac{(1+B)s^2 + (2\sigma + B\sigma + C)s + (\sigma^2+\omega_d^2)}{s[(s+\sigma)^2+\omega_d^2]}\end{aligned}$$

From the s^2 term: $B = -1$. From the s term: $2\sigma + B\sigma + C = 0 \Rightarrow C = -\sigma$. Thus:

$$\frac{G(s)}{s} = \frac{1}{s} - \left(\frac{s+\sigma}{(s+\sigma)^2+\omega_d^2} + \frac{\sigma}{\omega_d}\frac{\omega_d}{(s+\sigma)^2+\omega_d^2}\right)$$

From the tables, we can compute the inverse Laplace transform:

$$y(t) = 1 - \left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right).$$

Now we use the fact that:

$$\begin{aligned}\sqrt{A^2+B^2}\cos(\theta-\phi) &= \sqrt{A^2+B^2}\cos\phi\cos\theta + \sqrt{A^2+B^2}\sin\phi\sin\theta \\ &= A\cos\theta + B\sin\theta\end{aligned}$$

if

$$A = \sqrt{A^2+B^2}\cos\phi \qquad \text{and} \qquad B = \sqrt{A^2+B^2}\sin\phi.$$

so that $\tan\phi = B/A$. Thus

$$y(t) = 1 - \sqrt{1 + \frac{\sigma^2}{\omega_d^2}}\Big(\cos(\omega_d t - \beta)\Big)$$

where

$$\beta = \tan^{-1}\left(\frac{\sigma}{\omega_d}\right)$$

as claimed.