The Johns Hopkins University
Department of Electrical and Computer Engineering
520.353 — Control Systems
Fall 2007

Problem Set Number 2

Due: Wednesday, September 26, 2007 in class.

Note: Class on September 26 will be in the ECE lab (Barton 123) and will present an introduction to Matlab and Simulink for use in Control Systems

1. The *Gompertz* model of tumor growth is governed by the nonlinear differential equation

$$\dot{x}(t) = -ax(t)\ln(bx(t)), \qquad x(0) = c.$$

where $x(t)$ is proportional to the number of cells in the tumor and a, b and $c > 0$ are parameters.

(a) Find the equilibrium.

(b) Linearize the differential equation about the equilibrium.

(c) Using Laplace transfers, find the effect of small initial condition deviations from the equilibrium (i.e. $c = x_0 + \epsilon$).

Solution. We find the equilibria by setting the derivative equal to zero:

$$0 = -ax_0 \ln(bx_0)$$

The two equilibria occur when $x_0 = 0$, but this is not feasible because of the logarithm, or when $x_0 = 1/b > 0$. To linearize the equation, define:

$$h(x, \dot{x}) = \dot{x}(t) + ax(t)\ln(bx(t)) = 0.$$

Then

$$\begin{aligned} 0 &\approx \left.\frac{\partial h}{\partial \dot{x}}\right|_{\text{o.p}} \Delta\dot{x} + \left.\frac{\partial h}{\partial x}\right|_{\text{o.p}} \Delta x \\ 0 &= 1 \times \Delta\dot{x} + \left.(a\ln(bx(t)) + a)\right|_{\text{o.p}} \Delta x \\ 0 &= 1 \times \Delta\dot{x} + (a\ln(bx_0) + a)\,\Delta x. \end{aligned}$$

If $x_0 = 1/b$, then the linear equation is

$$\Delta\dot{x} = -a\Delta x$$

Using Laplace transforms:

$$s\Delta X(s) + a\Delta X(s) = \Delta x(0) = \epsilon \Rightarrow \Delta X(s) = \frac{\epsilon}{s+a}$$

which, in the time domain, is

$$\Delta x(t) = \epsilon e^{-at}, \qquad t \geq 0.$$

Note that this implies that

$$x(t) = x_0 + \Delta x(t) = \frac{1}{b} + \epsilon e^{-at}.$$

2. Linearize the equation

$$\dot{x}(t) = a\frac{u(t)}{x(t)} - cx(t), \qquad x(0) = 0,$$

and find the transfer function from u to x, where the constants $a > 0$ and $c \geq 0$. Use the operating point $x_0 = 1$.

Solution. At the equilibrium,

$$u_0 = \frac{c}{a}x_0^2 = \frac{c}{a} \geq 0.$$

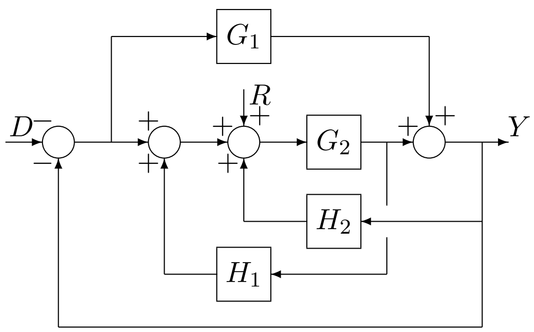
The linearized equation is

$$\Delta\dot{x}(t) = \left(\frac{a}{x_0}\right)\Delta u(t) - \left(\frac{au_0}{x_0^2} + c\right)\Delta x(t).$$

Using LT, we get the transfer function:

$$\frac{\Delta X(s)}{\Delta U(s)} = \frac{a/x_0}{s + au_0/x_0^2 + c} = \frac{a}{s + 2c}.$$

3. Find the transfer functions from R and D to Y for the following system:



Solution. Following the procedure taught in class, we write four transfer functions at the output of the summers:

$$\begin{aligned} X_1 &= -D - Y \\ X_2 &= X_1 + G_2H_1X_3 \\ X_3 &= R + X_2 + H_2Y \\ Y &= G_1X_1 + G_2X_3 \end{aligned}$$

Which can be written as:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & -1 & G_2H_1 & 0 \\ 0 & 1 & -1 & H_2 \\ G_1 & 0 & G_2 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ Y \end{bmatrix} = \begin{bmatrix} -D \\ 0 \\ -R \\ 0 \end{bmatrix}$$

From Cramer's rule:

$$Y = \frac{\det B}{\det A}, \quad A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & -1 & G_2H_1 & 0 \\ 0 & 1 & -1 & H_2 \\ G_1 & 0 & G_2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & -D \\ 1 & -1 & G_2H_1 & 0 \\ 0 & 1 & -1 & -R \\ G_1 & 0 & G_2 & 0 \end{bmatrix}$$

$$
\begin{aligned}
\det A &= \det \begin{bmatrix} -1 & G_2H_1 & 0 \\ 1 & -1 & H_2 \\ 0 & G_2 & -1 \end{bmatrix} - \det \begin{bmatrix} 1 & -1 & G_2H_1 \\ 0 & 1 & -1 \\ G_1 & 0 & G_2 \end{bmatrix} \\
&= -\det \begin{bmatrix} -1 & H_2 \\ G_2 & -1 \end{bmatrix} - \det \begin{bmatrix} G_2H_1 & 0 \\ G_2 & -1 \end{bmatrix} - \left(\det \begin{bmatrix} 1 & -1 \\ 0 & G_2 \end{bmatrix} + G_1 \det \begin{bmatrix} -1 & G_2H_1 \\ 1 & -1 \end{bmatrix} \right) \\
&= -1 + G_2H_2 + G_2H_1 - (G_2 + G_1(1 - G_2H_1)) \\
&= G_1G_2H_1 + G_2H_1 + G_2H_2 - G_1 - G_2 - 1
\end{aligned}
$$

and

$$
\begin{aligned}
\det B &= \det \begin{bmatrix} -1 & G_2H_1 & 0 \\ 1 & -1 & -R \\ 0 & G_2 & 0 \end{bmatrix} + D \det \begin{bmatrix} 1 & -1 & G_2H_1 \\ 0 & 1 & -1 \\ G_1 & 0 & G_2 \end{bmatrix} \\
&= R \det \begin{bmatrix} -1 & G_2H_1 \\ 0 & G_2 \end{bmatrix} + D \left(\det \begin{bmatrix} 1 & -1 \\ 0 & G_2 \end{bmatrix} + G_1 \det \begin{bmatrix} -1 & G_2H_1 \\ 1 & -1 \end{bmatrix} \right) \\
&= -G_2R + D\left(G_1 + G_2 - G_1G_2H_1\right)
\end{aligned}
$$

Now,

$$
\begin{aligned}
G_{YD} &= \frac{Y}{D}, \qquad \text{with } R = 0 \\
&= \frac{G_2 + G_1(1 - G_2H_1)}{G_1G_2H_1 + G_2H_1 + G_2H_2 - G_1 - G_2 - 1} \\
G_{YR} &= \frac{Y}{R}, \qquad \text{with } D = 0 \\
&= \frac{-G_2}{G_1G_2H_1 + G_2H_1 + G_2H_2 - G_1 - G_2 - 1}
\end{aligned}
$$