The Johns Hopkins University
Department of Electrical and Computer Engineering

520.454 — Control Systems Design — Spring 2007

Problem Set #0 — Solutions

Problems

1. Suppose that the input to a linear, time-invariant system was

$$u(t) = e^t \sin(2t), \qquad t \geq 0$$

and this resulted in the output

$$y(t) = e^{\alpha t}\cos(2t) + e^t \sin(2t) + e^{-3t}, \qquad t \geq 0.$$

(a) Let $\alpha = 1$. Find the transfer function of the system. Is the system bounded-input-bounded-output stable?

(b) Let $\alpha = 0$. Find the transfer function of the system. Is the system bounded-input-bounded-output stable in this case?

Solution. First compute the Laplace transform of both input and output:

$$U(s) = \frac{2}{(s-1)^2+4}, \quad Y(s) = \frac{s-\alpha}{(s-\alpha)^2+4} + \frac{2}{(s-1)^2+4} + \frac{1}{s+3}$$

The transfer function is

$$\begin{aligned}\frac{Y(s)}{U(s)} &= \frac{\frac{s-\alpha}{(s-\alpha)^2+4} + \frac{2}{(s-1)^2+4} + \frac{1}{s+3}}{\frac{2}{(s-1)^2+4}} \\ &= \frac{(s^2+(3-\alpha)s-3\alpha)(s^2-2s+5)+(s^2-2\alpha s+4+\alpha^2)[s^2+11]}{2(s+3)[(s-\alpha)^2+4]} \\ &= \frac{2s^4+[1-3\alpha]s^3+[14-\alpha+\alpha^2]s^2+[15-21\alpha]s+[44-15\alpha+11\alpha^2]}{2(s+3)([s-\alpha)^2+4]}\end{aligned}$$

(a) Suppose that $\alpha = 1$. Then the numerator can be factored as:

$$2s^4 - 2s^3 + 14s^2 - 6s + 40 = 2(s^2-2s+5)(s^2+s+4)$$

so that

$$\frac{Y(s)}{U(s)} = \frac{s^2+s+4}{s+3}$$

Note that even though the poles are in the left-hand plane, the system is not BIBOS because it is not proper.

(b) When $\alpha = 0$, the transfer function is

$$\frac{Y(s)}{U(s)} = \frac{2s^4+s^3+14s^2+15s+44}{2(s+3)(s^2+4)}$$

which is not stable because it has both poles on the imaginary axis (at $\pm 2j$) as is not proper.

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2. Let

$$k(PCF)(s) = k\frac{(s+1)^2+1}{s(s^2-1)}$$

(a) Draw the Nyquist diagram for $(PCF)(s)$.

(b) Use the Nyquist stability criterion to determine for which range of k the feedback system is internally stable.

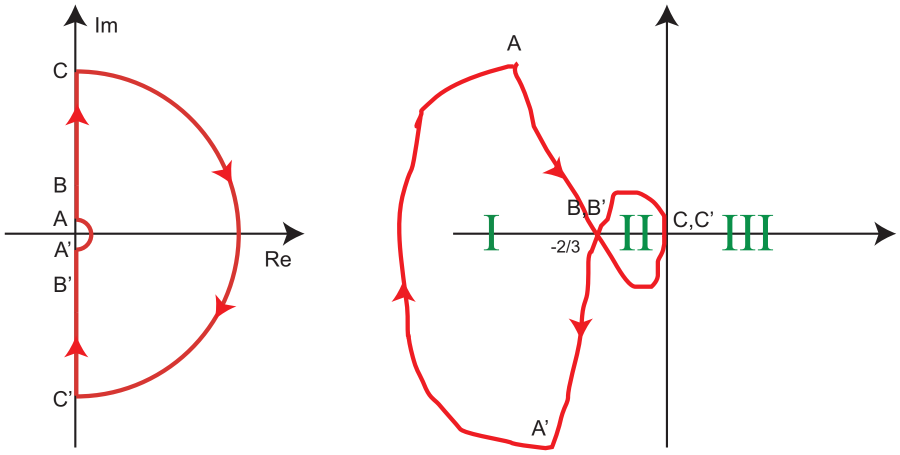
Solution. When $s = j\omega$ we have

$$\left.\frac{(s+1)^2+1}{s(s^2-1)}\right|_{s=j\omega} = \left.\frac{s^2+2+2s}{s(s^2-1)}\right|_{s=j\omega}$$
$$= \frac{2-\omega^2+2j\omega}{-j\omega(1+\omega^2)}$$
$$= \underbrace{\frac{-2}{1+\omega^2}}_{\text{Real part}} + j\underbrace{\frac{2-\omega^2}{\omega(1+\omega^2)}}_{\text{Imag. part}}$$

Note that:

if $\omega = 0^+$,	Real part is -2,	Imaginary part is $+\infty$;
if $\omega = \sqrt{2}$,	Real part is $-2/3$,	Imaginary part is 0;
if $\omega \uparrow \infty$,	Real part goes to 0^-,	Imaginary part goes to 0^-.

which leads to the following Nyquist plot:



Note that, in region I, there is one clockwise encirclement; in region II there is one CCW encirclement; in region III there are no encirclements. Because the open loop plant has one pole is the open right-hand plane, we need one CCW encirclement. I.e., $-1/k$ must be in region II:

$$-2/3 < \frac{-1}{k} < 0 \Longrightarrow k > 1.5$$

which you can confirm using the Routh-Hurwitz criterion.

3. Consider the standard closed-loop system with

$$P(s) = \frac{1}{s+1}, \qquad C(s) = K_1 + \frac{K_2}{s}, \qquad F(s) = 1.$$

Find gains K_1 and K_2 so that the closed-loop poles lie in the left half-plane $\operatorname{Re} s < -1$ and the steady-state error due to a unit ramp is less than 10%.

Solution. The characteristic polynomial is

$$\Delta(s) = s(s+1) + sK_1 + K_2 = s^2 + [1+K_1]s + K_2$$

To place the poles in the $\operatorname{Re} s < -1$ left half-plane, we make sure that

$$\Delta(s-1) = s^2 + [K_1 - 1]s + K_2 - K_1$$

is stable. But, from the Routh-Hurwitz criterion (for a second order system this is trivial) this is so iff

$$K_2 > K_1 > 1$$

For the ramp response condition, we note that the tracking error due to a unit ramp is

$$E(s) = \frac{1}{1+PC}R(s) = \frac{s(s+1)}{\Delta(s)}\frac{1}{s^2}$$

Because $\Delta(s)$ has poles in the left-hand plane, we can use the final value theorem:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s+1}{\Delta(s)} = \frac{1}{\Delta(0)} = \frac{1}{K_2}$$

Thus, we need $K_2 > 10$.

So any pair (K_1, K_2) satisfying $K_2 > 10$ and $K_2 > K_1 > 1$ will do.

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4. Consider the system

$$\begin{aligned} \dot{x} &= Ax + Bu, \qquad x(0) = x_0 \\ y &= Cx \end{aligned}$$

where

$$A = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix},$$

(a) Find e^{At}.

(b) Is the system stable? Asymptotically stable? Bounded-input-bounded-output stable?

(c) Suppose that $u \equiv 0$. Find $y(t)$ for $t \geq 0$.

Solution.

(a) Note that the A matrix is block-diagonal, so that

$$e^{At} = \begin{bmatrix} e^{A_1 t} & 0 \\ 0 & e^{A_2 t} \end{bmatrix}$$

where

$$A_1 = \begin{bmatrix} -4 & 0 \\ 1 & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

We can obtain the 2×2 matrix exponentials using the Laplace transform:

$$\begin{aligned}
e^{A_1 t} &= \mathcal{L}^{-1}(sI - A_1)^{-1} \\
&= \mathcal{L}^{-1}\begin{bmatrix} 1/(s+4) & 0 \\ 1/[(s+1)(s+4)] & 1/(s+1) \end{bmatrix} \\
&= \mathcal{L}^{-1}\begin{bmatrix} 1/(s+4) & 0 \\ \frac{1}{3}[1/(s+1) - 1/(s+4)] & 1/(s+1) \end{bmatrix} \\
&= \begin{bmatrix} e^{-4t} & 0 \\ \frac{1}{3}\left(e^{-t} - e^{-4t}\right) & e^{-t} \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
e^{A_2 t} &= \mathcal{L}^{-1}(sI - A_2)^{-1} \\
&= \mathcal{L}^{-1}\begin{bmatrix} s/(s^2+1) & 1/(s^2+1) \\ -1/(s^2+1) & s/(s^2+1) \end{bmatrix} \\
&= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}
\end{aligned}$$

(b) The system is stable (because all the eigenvalues are in $\operatorname{Re} s \le 0$, and the two eigenvalues on the imaginary axis are non-repeating, so that the Jordan block associated with them has to be one-dimensional). The system is not asymptotically stable, because of those imaginary axis eigenvalues. To determine whether it is BIBOS, we need to compute the transfer function:

$$G(s) = C(sI - A)^{-1}B = \begin{bmatrix} 1 & 0 \end{bmatrix}(sI - A_1)^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \end{bmatrix}(sI - A_2)^{-1}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$$

which is clearly UBIBOS.

(c) We have:

$$\begin{aligned}
y(t) &= Ce^{At}x_0 \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix}(sI - A_1)^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \end{bmatrix}(sI - A_2)^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} e^{-4t} & 0 \\ \frac{1}{3}\left(e^{-t} - e^{-4t}\right) & e^{-t} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= e^{-4t} + \cos t.
\end{aligned}$$

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5. For the system,

$$\begin{aligned}
\dot{x} &= Ax + Bu, \qquad x(0) = x_0 \\
y &= Cx
\end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \alpha \end{bmatrix}, \quad C = \begin{bmatrix} \beta & 1 & 0 & 0 \end{bmatrix}$$

(a) Determine what values of α make the system controllable.

(b) Determine what values of β make the system observable.

(c) When is the realization minimal?

Solution. We have

(a)

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 0 & 1 & \alpha-1 & 1-\alpha \\ 0 & \alpha & 1-\alpha & \alpha-1 \\ 1 & \alpha-1 & 1-\alpha & 0 \\ \alpha & 1-\alpha & \alpha-1 & 0 \end{bmatrix}$$

Now, working with the determinants leads to the following simplifications

$$\begin{aligned}
\det \mathcal{C} &= (1-\alpha)\det \begin{bmatrix} 0 & 1 & \alpha-1 & 1 \\ 0 & \alpha & 1-\alpha & -1 \\ 1 & \alpha-1 & 1-\alpha & 0 \\ \alpha & 1-\alpha & \alpha-1 & 0 \end{bmatrix} && \text{(from fourth column)} \\
&= (1-\alpha)^2 \det \begin{bmatrix} 0 & 1 & -1 & 1 \\ 0 & \alpha & 1 & -1 \\ 1 & \alpha-1 & 1 & 0 \\ \alpha & 1-\alpha & -1 & 0 \end{bmatrix} && \text{(from third column)} \\
&= (1-\alpha)^2 \det \begin{bmatrix} 0 & 1 & -1 & 1 \\ 0 & \alpha & 1 & -1 \\ 1 & \alpha-1 & 1 & 0 \\ 1+\alpha & 0 & 0 & 0 \end{bmatrix} && \text{(adding 3rd row to the 4th)} \\
&= (1-\alpha)^2 \det \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & \alpha & 1 & 0 \\ 1 & \alpha-1 & 1 & 1 \\ 1+\alpha & 0 & 0 & 0 \end{bmatrix} && \text{(adding 3rd column to the 4th)} \\
&= -(1-\alpha)^2 \det \begin{bmatrix} 0 & 1 & -1 \\ 0 & \alpha & 1 \\ 1+\alpha & 0 & 0 \end{bmatrix} \\
&= -(1-\alpha)^2(1+\alpha)\det \begin{bmatrix} 1 & -1 \\ \alpha & 1 \end{bmatrix} \\
&= -(1-\alpha)^2(1+\alpha)^2
\end{aligned}$$

so that controllability is lost when $\alpha \pm 1$.

(b) Similarly

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} \beta & 1 & 0 & 0 \\ 0 & 0 & \beta & 1 \\ 1-\beta & \beta-1 & 1-\beta & \beta-1 \\ 0 & 0 & 1-\beta & \beta-1 \end{bmatrix}$$

and

$$\begin{aligned}
\det \mathcal{O} &= (1-\beta)\det\begin{bmatrix} \beta & 1 & 0 & 0 \\ 0 & 0 & \beta & 1 \\ 1-\beta & \beta-1 & 1-\beta & \beta-1 \\ 0 & 0 & 1 & -1 \end{bmatrix} && \text{(from fourth row)} \\
&= (1-\beta)^2\det\begin{bmatrix} \beta & 1 & 0 & 0 \\ 0 & 0 & \beta & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} && \text{(from third row)} \\
&= (1-\beta)^2\det\begin{bmatrix} \beta+1 & 1 & 0 & 0 \\ 0 & 0 & \beta & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} && \text{(adding 2nd column to 1st)} \\
&= (1-\beta)^2(\beta+1)\det\begin{bmatrix} 0 & \beta & 1 \\ -1 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \\
&= (1-\beta)^2(\beta+1)\det\begin{bmatrix} \beta & 1 \\ 1 & -1 \end{bmatrix} \\
&= -(1-\beta)^2(\beta+1)^2
\end{aligned}$$

so that the system is observable iff $\beta \neq \pm 1$.

(c) The realization is minimal iff it is observable and controllable. I.e. when $\alpha \neq \pm 1$ and $\beta \neq \pm 1$.