Notes for Signals and Systems

8.4 CT LTI Frequency Response and Filtering

We can combine the Fourier series representation for periodic signals with the eigenfunction property for stable LTI systems to represent system responses to periodic input signals. Suppose the system has unit-impulse response $h(t)$. Since we will be considering different frequencies, it is convenient to change our earlier notation. Rather than think of a fixed frequency, ω_o, we think of frequency as a variable, ω, and define the *frequency response function* of the system by

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\, dt$$

Of course, the stability assumption guarantees that $H(\omega)$ is well defined for all ω. Then given a periodic input signal described, at least approximately, by the Fourier series expression

$$x(t) = \sum_{k=-K}^{K} X_k e^{jk\omega_o t}$$

linearity and the eigenfunction property give the output expression

$$y(t) = \sum_{k=-K}^{K} H(k\omega_o)X_k\, e^{jk\omega_o t}$$

Letting $Y_k = H(k\omega_o)X_k$, and noting the conjugacy property that $H(-\omega) = H^*(\omega)$, we see that the Y_k coefficients satisfy $Y_{-k} = Y_k^*$. This leads to the conclusion that the Y_k's are Fourier series coefficients for the periodic output signal. This expression for $y(t)$ can be converted to various real forms in the usual way. Of course, the output signal typically has the same fundamental frequency as the input signal, though not always since the frequency response function can be zero at particular frequencies.

This property also carries over to the case of causal, stable LTI systems with “right-sided periodic” input signals. Namely, the steady-state response is periodic and is as described above.

We can consider the magnitude of the frequency response function as a frequency-dependent gain of the system. That is, $|H(k\omega_o)|$ is the gain of the system at frequency $k\omega_o$, the factor by which the amplitude of the k^{th} harmonic of the input signal is increased or decreased. This is the basis of frequency selective filtering, where an LTI system is designed to have desired effects on the frequency components of the input signal. To show the filtering properties of a system, we often display a plot of the magnitude of the frequency response function, $|H(\omega)|$, vs. ω.

Example Consider again the *R*-*C* circuit in Section 6.6, with $R = 1, C = 1$. The unit-impulse response is

$$h(t) = \frac{1}{RC} e^{-\frac{1}{RC}t} u(t) = e^{-t} u(t)$$

Therefore the frequency response function for the circuit is

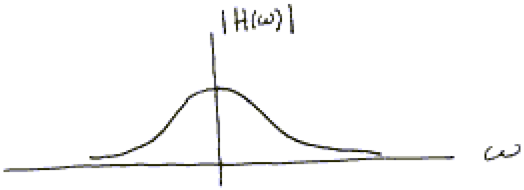
$$H(\omega) = \int_{-\infty}^{\infty} e^{-t} u(t) e^{-j\omega t} dt = \int_{0}^{\infty} e^{-(1+j\omega)t} dt$$

$$= \frac{1}{1+j\omega}$$

Since

$$|H(\omega)| = \frac{1}{\sqrt{1+\omega^2}}$$

it is straightforward to sketch the magnitude of the frequency response function:



Clearly the circuit acts as a low-pass filter, and high-frequency input signal components are attenuated much more than low-frequency components. To be specific, suppose

$$x(t) = 1 + \cos(t) + \cos(30t)$$

Since

$$x(t) = \mathrm{Re}\{e^{j0t}\} + \mathrm{Re}\{e^{jt}\} + \mathrm{Re}\{e^{j30t}\}$$

linearity and the eigenfunction property can be used to write the response as

$$y(t) = \mathrm{Re}\{H(0)e^{j0t}\} + \mathrm{Re}\{H(1)e^{jt}\} + \mathrm{Re}\{H(30)e^{j30t}\}$$

Then the computations

$$H(0) = 1, \quad H(1) = \frac{1}{\sqrt{2}} e^{-j\pi/4}, H(30) \approx \frac{1}{30} e^{-j\pi/2}$$

give

$$y(t) \approx 1 + \frac{1}{\sqrt{2}} \cos(t - \pi/4) + \frac{1}{30} \cos(30t - \pi/2)$$