Notes for Signals and Systems

8.3 Fourier Series Interpretations of Operations on Signals

Periodic signals are determined, to desired accuracy in terms of integral square error, by knowledge of the fundamental frequency, ω_o, and a suitable number of the complex-form Fourier series coefficients, $X_k\ ,\quad k = 0, \pm 1, \ldots, \pm K$. Thus the time-domain view of periodic signals is complemented by a "frequency domain" view, namely, the coefficients of various harmonic frequencies that make up the signal. This raises the possibility of performing or interpreting operations on signals by performing or interpreting operations on the frequency domain representation, that is, on the Fourier series coefficients

We will not go through a long list of operations, since this topic will reappear when we consider a more general frequency-domain viewpoint that includes aperiodic signals as well. However we consider a few examples.

Example Given a signal

$$x(t) = \sum_{k=-K}^{K} X_k e^{jk\omega_o t}$$

suppose a new signal is formed by amplitude transformation, $\hat{x}(t) = ax(t) + b$, where $a \neq 0$ and b are real constants. It is clear that $\hat{x}(t)$ is periodic, with the same fundamental period/frequency as $x(t)$, and indeed it is easy to determine the Fourier series coefficients of $\hat{x}(t)$ by inspection. We simply write

$$\hat{x}(t) = \sum_{k=-K}^{K} \hat{X}_k e^{jk\omega_o t} = b + a \sum_{k=-K}^{K} X_k e^{jk\omega_o t}$$

and conclude that

$$\hat{X}_k = \begin{cases} aX_0 + b\,, & k = 0 \\ aX_k\,, & k \neq 0 \end{cases}$$

This approach relies on the fact that the terms in a Fourier series for a periodic signal are unique, a fact that should be clear since each coefficient is determined independently of the others. However, a safer approach, especially for more complicated operations, is to begin with the expression for the Fourier series coefficients of the new signal, and relate it to the expression for coefficients of the original signal.

Example Suppose $\hat{x}(t) = x(at)$, where a is a nonzero constant. Then $\hat{x}(t)$ is periodic with fundamental period $\hat{T}_o = T_o / |\,a\,|$ and fundamental frequency $\hat{\omega}_o = |\,a\,|\,\omega_o$. The complex-form Fourier series coefficients are given by

$$\hat{X}_k = \frac{1}{\hat{T}_o} \int_0^{\hat{T}_o} \hat{x}(t) e^{-jk\hat{\omega}_o t}\, dt = \frac{|a|}{T_o} \int_0^{T_o/|a|} x(at) e^{-jk|a|\omega_o t}\, dt$$

To proceed, we need to separate the cases of positive and negative a. If $a < 0$, that is, $a = -|\,a\,|$, then the change of integration variable from t to $\tau = at = -|\,a\,|\,t$ gives

$$\hat{X}_k = \frac{|a|}{T_o} \int_0^{-T_o} x(\tau) e^{-jk|a|\omega_o \tau / a} \frac{d\tau}{-|a|} = \frac{1}{T_o} \int_{-T_o}^{0} x(\tau) e^{jk\omega_o \tau} \, d\tau$$

$$= \frac{1}{T_o} \int_{-T_o}^{0} x(\tau) \left(e^{-jk\omega_o \tau} \right)^* d\tau = X_k^* = X_{-k}$$

It is left as an exercise to show that for $a > 0$ a somewhat different result is obtained, namely

$$\hat{X}_k = X_k$$

That is, time scale by a positive constant leaves the Fourier series coefficients unchanged, though of course the fundamental frequency is changed. On the other hand, as a particular example, time scale by $a = -1$, which is time reversal, leaves the fundamental frequency unchanged, and the magnitude spectrum of the signal unchanged, but changes the phase spectrum.