

Notes for Signals and Systems

6.6 CT LTI Systems Described by Linear Differential Equations

Systems described by constant-coefficient, linear differential equations are LTI systems. However, while stating this fact it is important to keep in mind that by LTI we mean “input-output linear” systems and that our default time interval is $-\infty < t < \infty$. Because of this setting, our treatment may not be as similar to other treatments as you might expect.

Consider a system where the input and output signals are related by

$$\dot{y}(t) + ay(t) = bx(t), \quad -\infty < t < \infty$$

where a and b are real constants. This is called a *first-order, constant-coefficient, linear differential equation*. Once $x(t)$ is specified, this can be viewed as an equation that must be solved for $y(t)$. It can be shown that there is only one solution, and we will demonstrate that this solution can be written as

$$y(t) = \int_{-\infty}^t e^{-a(t-\tau)} bx(\tau) d\tau$$

The demonstration involves substituting into the differential equation, and proceeds in an elementary fashion by writing

$$y(t) = e^{-at} \int_{-\infty}^t e^{a\tau} bx(\tau) d\tau$$

In this form, the calculation of $\dot{y}(t)$ is a simple matter of the product rule, and the fundamental theorem of calculus. Indeed,

$$\begin{aligned} \dot{y}(t) &= -ae^{-at} \int_{-\infty}^t e^{a\tau} bx(\tau) d\tau + e^{-at} e^{at} bx(t) \\ &= -ay(t) + bx(t) \end{aligned}$$

and the solution is verified.

By inserting the appropriate unit-step function, we can write $y(t)$ in the form

$$y(t) = \int_{-\infty}^{\infty} be^{-a(t-\tau)} u(t-\tau) x(\tau) d\tau$$

and it is clear that the differential equation describes an LTI system with unit-impulse response

$$h(t) = be^{-at} u(t)$$

Remark It is interesting to show directly that this impulse response satisfies the differential equation (for all t) when $x(t) = \delta(t)$. The verification involves using generalized calculus to compute

$$\dot{h}(t) = -bae^{-at} u(t) + be^{-at} \delta(t) = -bae^{-at} u(t) + b\delta(t)$$

Then it is easy to see that

$$\dot{h}(t) + ah(t) = b\delta(t), \quad -\infty < t < \infty$$

From the form of the unit-impulse response, $h(t)$, it follows that the LTI system described by the first-order linear differential equation is causal and is not memoryless. The system is stable if and only if $a > 0$.

For a second-order, constant-coefficient, linear differential equation,

$$\ddot{y}(t) + a_1\dot{y}(t) + a_0y(t) = bx(t)$$

and also for higher-order linear differential equations, the situation is similar to the first-order case. Such equations describe causal LTI systems. However it is more difficult to compute the unit-impulse response, and to characterize stability properties in terms of the coefficients of the differential equation.

Right-Sided Setting In other courses you may have encountered linear differential equations defined for $t \geq 0$, with initial conditions specified at $t = 0^-$. For example, in the first-order case, consider

$$\dot{y}(t) + ay(t) = bx(t), \quad t \geq 0$$

with $y(0^-)$ and $x(t), t \geq 0$, specified. This setting can be embedded into our framework by considering the input signal to be zero for $t < 0$. Then, by causality, the output signal is zero for $t < 0$, and in particular, $y(0^-)$ must be zero. (Recall that if the input signal to an LTI system is zero for all t , then the output signal must be zero for all t .) Put another way, a constant-coefficient, linear differential equation with right-sided input signal describes an LTI system if and only if all initial conditions are zero.

Example Suppose a voltage signal, $x(t)$, is applied to the terminals of a series R - C circuit shown below, and the output signal of interest, $y(t)$, is the voltage across the capacitor, C .



Kirchhoff's voltage law gives the circuit description as a first-order differential equation

$$\dot{y}(t) + \frac{1}{RC} y(t) = \frac{1}{RC} x(t), \quad -\infty < t < \infty$$

This describes an LTI system with unit-impulse response

$$h(t) = \frac{1}{RC} e^{-\frac{1}{RC}t} u(t)$$

If we are interested in the response of this system to sinusoidal inputs with frequency ω_o , we consider the input signal

$$x(t) = e^{j\omega_o t}, \quad -\infty < t < \infty$$

and compute

$$\begin{aligned}
 H(\omega_o) &= \int_{-\infty}^{\infty} h(\tau) e^{-j\omega_o \tau} d\tau = \int_0^{\infty} \frac{1}{RC} e^{-\frac{1}{RC}\tau} e^{-j\omega_o \tau} d\tau \\
 &= \frac{-1}{1+jRC\omega_o} e^{-\left(\frac{1}{RC}+j\omega_o\right)\tau} \Bigg|_0^{\infty} \\
 &= \frac{1}{1+jRC\omega_o}
 \end{aligned}$$

(Notice that the implicit assumption that R and C are positive is crucial in the evaluation of the integral. This is the stability requirement – with positive R and C , $h(t)$ is absolutely integrable.) Thus the response to the phasor input signal is

$$y(t) = \frac{1}{1+jRC\omega_o} e^{j\omega_o t}, \quad -\infty < t < \infty$$

From this basic fact, we can extract the response to various sinusoidal input signals. For example, if the voltage input signal is

$$x(t) = \cos(\omega_o t) u(t)$$

Then the steady-state response of the circuit can be written as

$$\begin{aligned}
 y_{ss}(t) &= \operatorname{Re} \left\{ \frac{1}{1+jRC\omega_o} e^{j\omega_o t} \right\} = \operatorname{Re} \left\{ \frac{1}{\sqrt{1+R^2C^2\omega_o^2}} e^{j(\omega_o t - \tan^{-1}(RC\omega_o))} \right\} \\
 &= \frac{1}{\sqrt{1+R^2C^2\omega_o^2}} \cos[\omega_o t - \tan^{-1}(RC\omega_o)]
 \end{aligned}$$

If the input frequency, ω_o , is large, then the steady-state voltage across the capacitor will be small. On the other hand, if the input frequency is small, then the steady-state response is similar in amplitude to the input signal. The phase angle of the response, relative to the input signal, also depends on the frequency. Furthermore, if the input signal is a linear combination of sinusoids at various frequencies, then the steady-state response will contain the same set of frequencies, but with the amplitudes and phase angles influenced according to $H(\omega_o)$ at the various values of ω_o . This is the basis of frequency-selective filtering.