- **Notes for Signals and Systems**

6.3 CT LTI System Properties

The input-output behavior of a continuous-time LTI system is described by its unit-impulse response, $h(t)$, via the convolution expression

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Therefore the input-output properties of an LTI system can be characterized in terms of properties of $h(t)$. The basic results are similar to the discrete-time case, though, as usual in continuous time, there are unmentioned technical assumptions to guarantee that integrals are defined, and so on.

- *Causal System* An LTI system is causal if and only if $h(t) = 0$ for $t < 0$, that is, if and only if $h(t)$ is right sided.

Sufficiency follows by inspection of the convolution expression. For necessity we use the facts that the response of an LTI system to the identically-zero input is the identically-zero output, and the unit-impulse input is nonzero only at $t = 0$. Therefore, if $h(t)$ is nonzero for some negative value of *t*, then the system response at that time depends on information about the impulse input at the later time, $t = 0$.

- *Memoryless System* An LTI system is memoryless if and only if $h(t) = 0$ for $t \neq 0$.

If $h(t) = 0$ for $t \neq 0$, then since

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

it follows that $y(t)$ can only depend on $x(t)$. On the other hand, if $h(t) \neq 0$ for $t = t_a \neq 0$, then the unit-impulse input, which is nonzero only at $t = 0$ yields a response that is nonzero at the nonzero time t_a. Thus the system is not memoryless.

Suppose $h(t)$ is nonzero at only one point in time. Then unless $h(t)$ is an impulse the response of the system to every input signal will be $y(t) = 0$ for all t. It follows from this discussion that a memoryless LTI system is characterized by an impulse response of the form $h(t) = b\,\delta(t)$, where *b* is a real constant.

- *Stable System* An LTI system is (bounded-input, bounded-output) stable if and only if the unit-impulse response is absolutely integrable. That is

$$\int_{-\infty}^{\infty} |\,h(t)\,|\,dt$$

is finite.

To prove this, suppose $x(t)$ is a bounded input, and $|x(t)| \leq M$, for all t. We use the fact that the absolute value of an integral with upper limit greater than lower limit is bounded by the integral of the absolute value of the integrand. This should be believable from the corresponding fact about sums. Then the absolute value of the output signal satisfies

$$|y(t)| \leq \int_{-\infty}^{\infty} |h(\tau)||x(t-\tau)|\, d\tau \leq M \int_{-\infty}^{\infty} |h(\tau)|\, d\tau \leq MK$$

for all t, and therefore the system is stable.

To prove that stability of the system implies absolute integrability of $h(t)$, we use the same sort of cleverness as in the discrete-time case. Consider the bounded input signal

$$x(t) = \begin{cases} 1, & h(-t) \geq 0 \\ -1, & h(-t) < 0 \end{cases}$$

Then the corresponding output signal, $y(t)$, is bounded, say by the constant K, for all t. In particular, at $t = 0$,

$$K \geq y(0) = \int_{-\infty}^{\infty} h(\tau)x(-\tau)\, d\tau = \int_{-\infty}^{\infty} |h(\tau)|\, d\tau$$

Thus $h(t)$ is absolutely integrable.

- *Invertible System* First note that the identity system in continuous time, $y(t) = x(t)$, has the unit impulse response $h(t) = \delta(t)$, and the inverse system for an LTI system must be an LTI system. Then we can make the following statement: An LTI system described by $h(t)$ is invertible if and only if there exists a signal $h_I(t)$ (the impulse response of the inverse system) such that

$$(h * h_I)(t) = \delta(t)$$

Such an $h_I(t)$ might not exist, and if it does, it might be difficult to compute. We will not pursue this further.