

Notes for Signals and Systems

5.6 DT LTI Systems Described by Linear Difference Equations

Systems described by constant-coefficient, linear difference equations are LTI systems. In exploring this fact, it is important to keep in mind that our default setting is that all signals are defined for $-\infty < n < \infty$. This brings about significant differences (!) with other treatments of difference equations.

Suppose we have a system whose input and output signals are related by

$$y[n] + ay[n-1] = bx[n], \quad -\infty < n < \infty$$

where a and b are real constants. This is called a *first-order, constant-coefficient, linear difference equation*. Given an input signal $x[n]$, this can be viewed as an equation that must be solved for $y[n]$, and we leave to other courses the argument that for each input signal there is a unique solution for the output signal. Instead, we simply make the claim that the solution is

$$y[n] = \sum_{k=-\infty}^n (-a)^{n-k} bx[k]$$

and verify this solution as follows. Using the assumed $y[n]$,

$$\begin{aligned} ay[n-1] &= a \sum_{k=-\infty}^{n-1} (-a)^{n-1-k} b x[k] \\ &= - \sum_{k=-\infty}^{n-1} (-a)^{n-k} b x[k] \end{aligned}$$

Therefore

$$\begin{aligned} y[n] + ay[n-1] &= \sum_{k=n}^n (-a)^{n-k} bx[k] \\ &= bx[n] \end{aligned}$$

Of course, the solution can be written as

$$y[n] = \sum_{k=-\infty}^{\infty} (-a)^{n-k} bu[n-k] x[k]$$

so it is clear that the difference equation describes an LTI system with the unit-pulse response

$$h[n] = (-a)^n bu[n]$$

That this is the unit-pulse response also can be verified directly, by showing that

$$h[n] + ah[n-1] = b\delta[n]$$

From the form of $h[n]$ it follows that a first-order, constant-coefficient, linear difference equation defines a causal LTI system. Furthermore the system is memoryless if and only if $a = 0$, and stable if and only if $|a| < 1$.

Results are similar for systems described by second-order, constant-coefficient, linear difference equations,

$$y[n] + a_1 y[n-1] + a_2 y[n-2] = bx[n], \quad -\infty < n < \infty$$

as well as higher order. That is, such equations describe causal, LTI systems. However, it is more difficult to compute the unit-pulse response, and to connect the stability property to the coefficients in the difference equation.

Right-Sided Setting Often we are interested in right-sided inputs, where the causality property of systems described by difference equations implies that the corresponding outputs are also right sided. This means that the difference equation need only be addressed for $n \geq 0$, though we must view the output as zero for negative values of n . Therefore the “initial conditions” are $y[-1] = y[-2] = \dots = 0$. Nonzero initial conditions, as considered in mathematics or other engineering courses dealing with difference equations, cannot arise in the context of LTI systems, for a consequence of linearity is that the identically zero input signal must yield the identically zero output signal. In summary, since we are focusing on systems whose input-output behavior is linear, we must require zero initial conditions in a right-sided setting.