Notes for Signals and Systems

5.3 DT LTI System Properties

Since the input-output behavior of a discrete-time LTI system is completely characterized by its unit-pulse response, $h[n]$, via the convolution expression

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

the input-output properties of the system can be characterized very precisely in terms of properties of $h[n]$.

- *Causal System* An LTI system is causal if and only if $h[n] = 0$ for $n < 0$, that is, if and only if $h[n]$ is right sided.

The proof of this is quite easy from the convolution expression. If the unit-pulse response is right sided, then the convolution expression simplifies to

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k]$$

and, at any value of n, the value of $y[n]$ depends only on the current and earlier values of the input signal. If the unit-pulse response is not right sided, then it is easy to see that the value of $y[n]$ at a particular n depends on future values of the input signal.

- *Memoryless System* An LTI system is memoryless if and only if $h[n] = c\delta[n]$, for some constant c. Again, a proof is quite easy to argue from the convolution expression.

- *Stable System* An LTI system is (bounded-input, bounded-output) stable if and only if the unit-pulse response is absolutely summable. That is,

$$\sum_{n=-\infty}^{\infty} |h[n]|$$

is finite.

To prove this, suppose $x[n]$ is a bounded input, that is, there is a constant M such that $|x[n]| \le M$ for all n. Then the absolute value of the output signal satisfies

$$|y[n]| = |\sum_{k=-\infty}^{\infty} h[k]x[n-k]| \le \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]|$$

$$\le M \sum_{k=-\infty}^{\infty} |h[k]|$$

Therefore, if the absolute summability condition holds, the output signal is bounded for any bounded input signal, and we have shown that the system is stable.

To prove that stability of the system implies absolute summability requires considerable cleverness. Consider the input $x[n]$ defined by

$$x[-n] = \begin{cases} 1, & h[n] \geq 0 \\ -1, & h[n] < 0 \end{cases}$$

Clearly $x[n]$ is a bounded input signal, and the corresponding response $y[n]$ at $n = 0$ is

$$y[0] = \sum_{k=-\infty}^{\infty} h[k]\, x[-k] = \sum_{k=-\infty}^{\infty} |\, h[k]\,|$$

Since the system is stable, $y[n]$ is bounded, and therefore $y[0]$ is bounded, and therefore the unit-pulse response is absolutely summable.

Example The system with unit-pulse response

$$h[n] = (0.5)^n u[n]$$

is a stable system since

$$\sum_{n=-\infty}^{\infty} |\, h[n]\,| = \sum_{n=0}^{\infty} (0.5)^n = \frac{1}{1-0.5} = 2$$

On the other hand, the system with unit-pulse response

$$h[n] = u[n-1]$$

is unstable.

- *Invertible System* There is no simple characterization of invertibility in terms of the the unit-pulse response. However, in particular examples it is sometimes possible to compute the unit-pulse response of the inverse system, $h_I[n]$ from the requirement

$$(h * h_I)[n] = \delta[n]$$

Example To compute an inverse of the running summer, that is, the LTI system with unit pulse response $h[n] = u[n]$, we must find $h_I[n]$ that satisfies

$$\sum_{k=-\infty}^{\infty} u[k]\, h_I[n-k] = \delta[n]$$

Simplifying the summation gives

$$\sum_{k=0}^{\infty} h_I[n-k] = \delta[n]$$

It is clear that we should take $h_I[n] = 0$, for $n < 0$. For $n = 0$ the condition is

$$\sum_{k=0}^{\infty} h_I[-k] = h_I[0] = 1$$

For $n = 1$ the condition is

$$\sum_{k=0}^{\infty} h_I[1-k] = h_I[1] + h_I[0] = 0$$

which gives $h_I[1] = -1$. Continuing for further values of *n*, it is clear that the condition is satisfied by taking all remaining values of $h_I[n]$ to be zero. Thus the inverse system has the unit pulse response

$$h_I[n] = \delta[n] - \delta[n-1]$$

Of course, it is easy to see that in general the output of this inverse system is the first difference of the input signal.