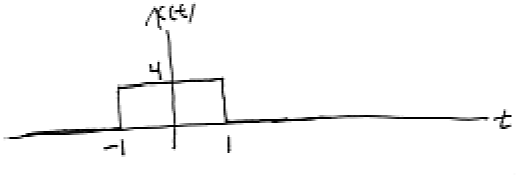
Notes for Signals and Systems

2.3 Linear Combinations of Singularity Signals and Generalized Calculus

For simple signals, that is, signals with uncomplicated wave shapes, it is convenient for many purposes to use singularity signals for representation and calculation.

Example The signal shown below can be written as a sum of step functions,

$$x(t) = 4u(t+1) - 4u(t-1)$$



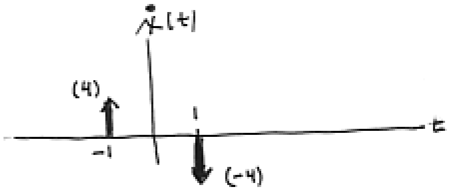
Another representation is

$$x(t) = 4u(t+1)u(1-t)$$

This uses the unit steps as "cutoff" functions, and sometimes this usage is advantageous. However for further calculation, representation as a linear combination usually is much simpler. Differentiation of the first expression for $x(t)$ gives, using our generalized notion of differentiation,

$$\dot{x}(t) = 4\delta(t+1) - 4\delta(t-1)$$

This signal is shown below.



The same result can be obtained by differentiating the "step cutoff" representation for $x(t)$, though usage of the product rule and interpretation of the final result makes the derivative calculation more difficult. That is,

$$\begin{aligned}\dot{x}(t) &= 4\delta(t+1)u(1-t) + 4u(t+1)\delta(1-t)\\ &= 4\delta(t+1) - 4\delta(1-t)\\ &= 4\delta(t+1) - 4\delta(t-1)\end{aligned}$$

(The first step makes use of the product rule for differentiation, the second step uses the signal-multiplication rule for impulses, and the last step uses the evenness of the impulse.)

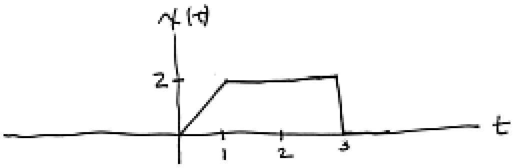
Of course, regardless of the approach taken, graphical methods easily verify

$$\int_{-\infty}^{t} \dot{x}(\tau)\, d\tau = x(t)$$

in this example. Note that the constant of integration is taken to be zero since it is known that the signal $x(t)$ is zero for $t < -1$.

Example The signal shown below can be written as

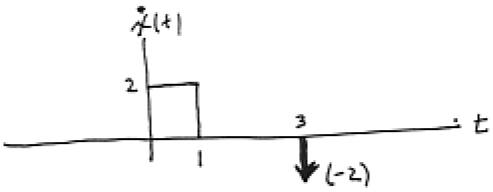
$$x(t) = 2r(t) - 2r(t-1) - 2u(t-3)$$



Again, the derivative is straightforward to compute,

$$\dot{x}(t) = 2u(t) - 2u(t-1) - 2\delta(t-3)$$

and sketch,



Graphical interpretations of differentiation support these computations.

While representation in terms of linear combinations of singularity signals leads to convenient shortcuts for some purposes, caution should be exercised. In the examples so far, well-behaved energy signals have been represented as linear combinations of signals that are power signals, singularity signals, and unbounded signals. This can introduce complications in some contexts.

Sometimes we use these generalized calculus ideas for signals are nonzero for infinite time intervals.

Example A right-sided cosine signal can be written as

$$x(t) = \cos(2t)\, u(t)$$

Then differentiation using the product rule, followed by the impulse multiplication rule, gives

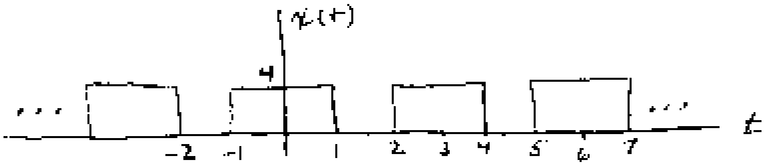
$$\begin{aligned}\dot{x}(t) &= -2\sin(2t)\, u(t) + \cos(2t)\, \delta(t) \\ &= -2\sin(2t)\, u(t) + \delta(t)\end{aligned}$$

You should graphically check that this is a consistent result, and that the impulse in $\dot{x}(t)$ is crucial in verifying the relationship

$$\int_{-\infty}^{t} \dot{x}(\tau)\, d\tau = x(t)$$

Example The periodic signal shown below can be written as

$$x(t) = \sum_{k=-\infty}^{\infty} \left[4u(t+1-3k) - 4u(t-1-3k)\right]$$



Then

$$\dot{x}(t) = \sum_{k=-\infty}^{\infty} \left[4\delta(t+1-3k) - 4\delta(t-1-3k)\right]$$

by generalized differentiation, and a sketch of $\dot{x}(t)$ is shown below.