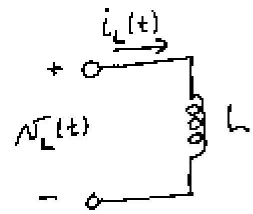
Notes for Signals and Systems

12.2 Circuits with Nonzero Initial Conditions

For circuit elements with nonzero initial stored energy, that is, nonzero initial conditions, we can develop Laplace transform equivalent circuits that represent the initial conditions as voltage or current sources.

For an inductor,



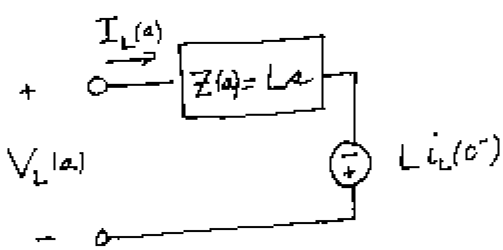
with initial current $i_L(0^-)$ in the indicated direction, the voltage-current relation in the time domain remains

$$v_L(t) = L\frac{di_L(t)}{dt}, \quad t \ge 0$$

However, the unilateral Laplace transform differentiation property, taking account of the initial condition, yields

$$\begin{aligned} V_L(s) &= L[sI_L(s) - i_L(0^-)] \\ &= LsI_L(s) - Li_L(0^-)] \end{aligned}$$

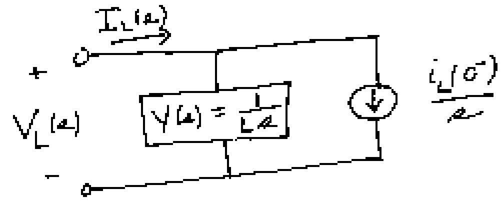
This corresponds to the transform equivalent circuit shown below, where the initial condition term is represented as a voltage source with appropriate polarity:



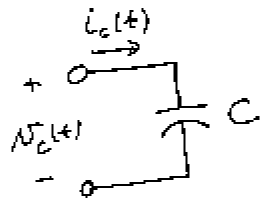
Of course, an alternate approach is to write

$$I_L(s) = \frac{1}{Ls}V_L(s) + \frac{i_L(0^-)}{s}$$

This leads to an admittance version of the transform equivalent, where the initial condition is represented as a current source with appropriate polarity:



Similar calculations for a capacitor are almost apparent. With an initial voltage $v_C(0^-)$, the capacitor with polarity as marked



is described by

$$C\,\dot{v}_C(t) = i_C(t)\,, \quad t \geq 0$$

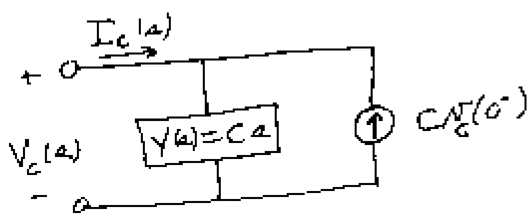
The Laplace transform differentiation property gives

$$C[sV_C(s) - v_C(0^-)] = I_C(s)$$

or

$$I_C(s) = CsV_C(s) - Cv_C(0^-)$$

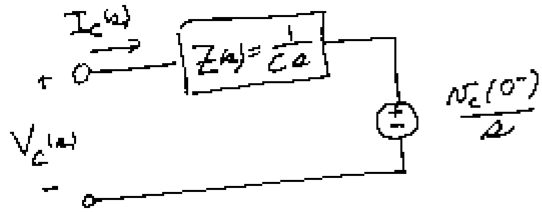
This corresponds to the transform equivalent circuit shown below, where the initial condition is represented by a current source.



An alternative is to write

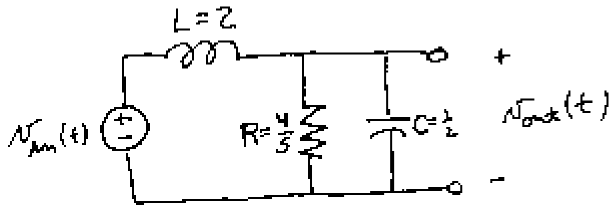
$$V_C(s) = \frac{1}{Cs} I_C(s) + \frac{v_C(0^-)}{s}$$

which corresponds to the circuit

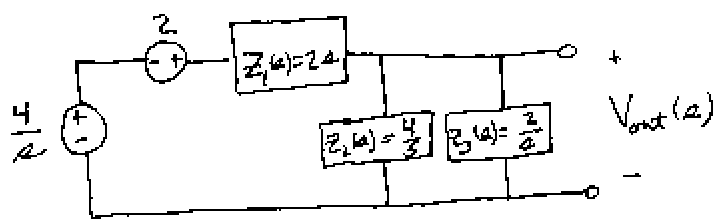


where a voltage source accounts for the initial condition.

Example Consider the circuit



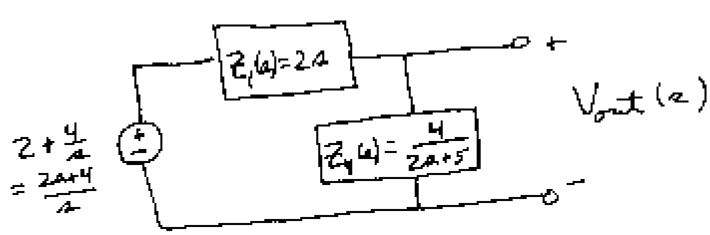
where the input voltage is $v_{in}(t) = 4u(t)$, the initial current in the inductor is $i_L(0^-) = 1$, and the initial voltage on the capacitor is zero. To compute the output, $v_{out}(t)$, we first sketch the transform equivalent circuit:



The two voltage sources can be combined, and impedances in parallel can be combined according to

$$Z_4(s) = \frac{Z_2(s) Z_3(s)}{Z_2(s) + Z_3(s)} = \frac{4}{2s+5}$$

This gives the equivalent circuit shown below



Now a straightforward voltage-divider calculation gives $V_{out}(s)$:

$$V_{out}(s) = \frac{\frac{4}{2s+5}}{2s + \frac{4}{2s+5}} \frac{2s+4}{s} = \frac{2s+4}{s^3 + \frac{5}{2}s^2 + s} = \frac{2}{s(s+\frac{1}{2})}$$

and partial fraction expansion leads to

$$v_{out}(t) = 4u(t) - 4e^{-\frac{1}{2}t}u(t)$$