Notes for Signals and Systems

11.4 Systems Described by Linear Differential Equations

Consider a system where the input and output signals are related by the first-order differential equation

$$\dot{y}(t) + ay(t) = bx(t)$$

Assuming that the input signal is right sided, and assuming that initial condition at $t = 0$ is zero, the output signal is right sided and the system is linear and time invariant. (In particular, since an LTI system with identically zero input must have identically zero output, the assumption of zero initial condition is important.)

In the setting of right-sided input and output signals, the system can be described in terms of unilateral Laplace transforms. Regardless of the values of the constants a and b, and in particular regardless of the stability property of the system, we can compute the Laplace transform of the unit impulse response

$$h(t) = be^{-at}u(t)$$

to obtain

$$H(s) = \frac{b}{s+a}$$

Rather than the term frequency response function, this is called the *transfer function* of the system, and in terms of the Laplace transforms $X(s)$ and $Y(s)$ of the right-sided input and output signals the system is described by

$$Y(s) = H(s)X(s)$$

Another approach is to equate the Laplace transforms of the left and right sides of the differential equation, and this approach has the advantage of not requiring knowledge of the unit-impulse response. Using the linearity and differentiation properties gives

$$(s+a)Y(s) = bX(s)$$

Thus, again, we obtain

$$\frac{Y(s)}{X(s)} = H(s) = \frac{b}{s+a}$$

If the input signal has a proper rational Laplace transform, then it is clear that the output signal has a strictly-proper rational Laplace transform. Therefore we can solve for the response to a wide class of input signals by the algebraic process of partial fraction expansion and table lookup.

Again, an advantage of the Laplace-transform approach in this unilateral setting is that systems with unbounded input signals and/or output signals, or systems that are unstable, can be treated, in contrast to the Fourier transform approach.

Example For the case where $a = -1$, $b = 1$ and where the input signal is

$$x(t) = e^{3t}u(t)$$

that is, an unstable system with unbounded input signal, we immediately obtain

$$Y(s) = \frac{1}{(s-1)(s-3)}$$

Partial fraction expansion easily leads to

$$y(t) = -\tfrac{1}{2}e^{t}u(t) + \tfrac{1}{2}e^{3t}u(t)$$

For systems described by higher order linear differential equations, again with unilateral input and output signals and zero initial conditions,

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_0 y(t) = b_{n-1}x^{(n-1)}(t) + \cdots + b_0 x(t)$$

it is straightforward to equate the Laplace transforms of the right and left sides to show that the corresponding transfer function is

$$H(s) = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

This is a strictly-proper rational function of s. Thus for input signals that have proper rational Laplace transforms, the output signal will have a proper rational Laplace transform, and the solution procedure for the output signal is again algebraic, though of course the roots of the denominator must be computed for the partial fraction expansion.