Notes for Signals and Systems

11.3 Inverse Unilateral Laplace Transform

Inspection of the Laplace transforms we have computed, or a table of transforms, indicates that the signals typically encountered have transforms that are strictly-proper rational functions. These are ratios of polynomials in s with the degree of the numerator polynomial less than the degree of the denominator polynomial. As might be expected from the Fourier-transform case, partial fraction expansion, followed by table lookup, is the main tool for computing the time signal corresponding to a given transform. (There is a more general inverse transform formula, but it involves line integrals in the complex plane and we will not make use of it.)

We illustrate the calculation of inverse transforms with two examples.

Example Given

$$X(s) = \frac{s^2 + s + 1}{s^2 + 1}$$

which is a proper, but not strictly-proper, rational function, we can divide the numerator by the denominator to write

$$X(s) = 1 + \frac{s}{s^2 + 1}$$

Using linearity of the Laplace transform, we can treat the terms separately. Partial fraction expansion of the second term gives

$$\frac{s}{s^2 + 1} = \frac{s}{(s+j)(s-j)} = \frac{1/2}{s+j} + \frac{1/2}{s-j}$$

From the table of transforms,

$$L^{-1}\left[\frac{s}{s^2 + 1}\right] = \tfrac{1}{2}e^{-jt}u(t) + \tfrac{1}{2}e^{jt}u(t) = \cos(t)\,u(t)$$

Therefore

$$x(t) = \delta(t) + \cos(t)\,u(t)$$

Another case that is straightforward to handle is when there are “delay factors” in the transform.

Example Given

$$X(s) = \frac{s\,e^{-2s} + e^{-s}}{s^2 + 1}$$

we can write

$$X(s) = e^{-2s}\,\frac{s}{s^2 + 1} + e^{-s}\,\frac{1}{s^2 + 1}$$

Using the linearity, delay, and derivative properties in conjunction with the previous example, we obtain

$$x(t) = \cos(t-2)\,u(t) + \sin(t-1)\,u(t-1)$$