Notes for Signals and Systems

10.7 Fourier Transform and LTI Systems Described by Differential Equations

If a system is described by a first-order, linear differential equation,

$$\dot{y}(t) + ay(t) = bx(t)\,, \quad -\infty < t < \infty$$

then from Section 6.6 we have that the system is linear and time invariant, and the unit-impulse response is given by

$$h(t) = be^{-at}u(t)$$

Therefore we can readily compute the frequency response function of the system,

$$H(\omega) = \frac{b}{a + j\omega}$$

However, this is valid only if the system is stable, that is, $a > 0$. (If $a < 0$, then the Fourier transform of $h(t)$ does not exist, and if $a = 0$, then the Fourier transform has a different form.) Because $H(\omega)$ is a strictly-proper rational function, if the input signal has a proper rational Fourier transform, computation of the response $y(t)$ is simply a matter of computing the inverse Fourier transform of $Y(\omega) = H(\omega)\,X(\omega)$ by partial-fraction expansion and table lookup. That is, for a large class of input signals, the response computation is completely algebraic.

More directly, we can express the relation between time signals in the differential equation as a relation between Fourier transforms. Letting

$$X(\omega) = F[x(t)]\,, \quad Y(\omega) = F[y(t)]$$

linearity and the differentiation property give

$$j\omega Y(\omega) + a\,Y(\omega) = b\,X(\omega)$$

This can be solved algebraically to obtain

$$Y(\omega) = \frac{b}{a + j\omega}\,X(\omega)$$

From this we recognize $H(\omega)$, and the obvious inverse Fourier transform gives $h(t)$, the unit-impulse response of the system. Again, this is valid only for $a > 0$, and a danger is that this condition is not apparent until the inverse Fourier transform is attempted. In other words, the stability condition is not explicit in the algebraic manipulations leading to the frequency response function.

It should be clear that this approach applies to higher-order, linear differential equations that correspond to *stable* systems. The frequency response function in such a case can be written in the form

$$H(\omega) = \frac{b_m(j\omega)^m + \cdots + b_1(j\omega) + b_0}{(j\omega)^n + a_{n-1}(j\omega)^{n-1} + \cdots + a_0}$$

where $m < n$. So, again, computation of the response to a large class of input signals is completely algebraic (though checking the stability condition is more subtle, and is omitted).

Example When the input signal has a Fourier transform that is not a proper rational function, the calculations become slightly more complicated and involve some recognition of combinations of terms. Suppose $a = 2, b = 1$, and the input signal is a unit step function. Then

$$Y(\omega) = \frac{1}{2 + j\omega}\left[\frac{1}{j\omega} + \pi\,\delta(\omega)\right] = \frac{1}{(j\omega)(2 + j\omega)} + \frac{\pi}{2 + j\omega}\;\delta(\omega)$$

$$= \frac{1/2}{j\omega} - \frac{1/2}{2 + j\omega} + \frac{\pi}{2}\delta(\omega)$$

Grouping together the first and last terms, table lookup gives the output signal as

$$y(t) = \tfrac{1}{2}u(t) - \tfrac{1}{2}e^{-2t}u(t)$$