

## Notes for Signals and Systems

### 10.2 Fourier Transform for Periodic Signals

If  $x(t)$  is  $T_o$ -periodic, then it is clear that the Fourier transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

does not exist in the usual sense, because of the failure of the integral to converge. However, we can take an indirect approach and use notions of generalized functions to extend the Fourier transform to periodic signals in a way that captures the Fourier series expression in a fashion consistent with ordinary Fourier transforms. The Fourier series expression

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_o t}$$

indicates that the key is to develop the Fourier transform of the complex signal

$$x(t) = e^{j\omega_o t}$$

One approach is to use the Special Property 2 in Section 2.2 again:

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} e^{j\omega_o t} e^{-j\omega t} dt = \int_{-\infty}^{\infty} e^{j(\omega_o - \omega)t} dt = 2\pi\delta(\omega_o - \omega) \\ &= 2\pi\delta(\omega - \omega_o) \end{aligned}$$

That this is reasonable is easily verified using the inverse Fourier transform, with evaluation by the sifting property.

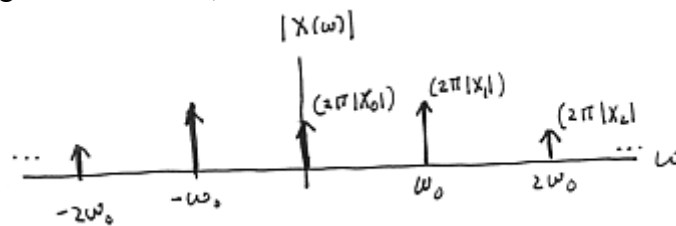
Following this calculation, we can compute the Fourier transform of a signal expressed by a Fourier series in a straightforward manner:

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_o t} e^{-j\omega t} dt \\ &= \sum_{k=-\infty}^{\infty} X_k \int_{-\infty}^{\infty} e^{jk\omega_o t} e^{-j\omega t} dt = \sum_{k=-\infty}^{\infty} 2\pi X_k \delta(\omega - k\omega_o) \end{aligned}$$

In words, to compute the Fourier transform of a periodic signal, first compute the Fourier series coefficients,  $X_k$ , and then simply substitute this data into the  $X(\omega)$  expression above.

Of course the Fourier transform expression is invertible by inspection, in the sense that the Fourier series for  $x(t)$  can be written by inspection from  $X(\omega)$ . From another viewpoint, we essentially have built the Fourier series into the transform. We need to reformat the notions of spectra of  $x(t)$  in consonance with this new framework, but that is easy. For any value of  $\omega$ , at most one summand in the expression for  $X(\omega)$  can be nonzero. Because the summands are non-overlapping in this sense, the magnitude of the sum is the sum of the magnitudes. Instead of a line plot, however, the magnitude

spectrum (or amplitude spectrum, if every  $X_k$  is real) becomes a plot of impulse functions, occurring at integer multiples of the fundamental frequency, labeled with the impulse-area magnitudes. That is,



The phase spectrum computes in a similar fashion, since the angle of a sum of non-overlapping terms is the sum of the angles. Therefore the phase spectrum is interpreted as a line plot of the angles of the areas of impulses vs frequency. That is, the phase spectrum has exactly the same form as in the context of Fourier series representations.

*Example* A couple of special cases are interesting to note, based on the Fourier transform of a phasor. First, for

$$x(t) = \cos(\omega_0 t) = \frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t}$$

we have

$$X(\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

Second, for  $x(t) = \sin(\omega_0 t)$ ,

$$X(\omega) = -j\pi\delta(\omega - \omega_0) + j\pi\delta(\omega + \omega_0)$$

*Example* Other approaches to computing the Fourier transform of periodic signals can give “correct,” but difficult to interpret results. For the signal

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - k)$$

we directly compute

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \delta(t - k) e^{-j\omega t} dt = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(t - k) e^{-j\omega t} dt \\ &= \sum_{k=-\infty}^{\infty} e^{-j\omega k} \end{aligned}$$

On the other hand, the recommended procedure is to compute the Fourier series data for  $x(t)$ , which easily yields  $\omega_0 = 2\pi$  and  $X_k = 1/(2\pi)$ , for all  $k$ . Then by inspection we obtain the Fourier transform

$$X(\omega) = \sum_{k=-\infty}^{\infty} \delta(\omega - k2\pi)$$

Needless to say, it is difficult to show by elementary means that these two expressions for  $X(\omega)$  are the same. In any case, we always prefer the second.