Notes for Signals and Systems

0.2 Background in Complex Arithmetic

We assume easy familiarity with the arithmetic of complex numbers. In particular, the polar form of a complex number c, written as

$$c = |c| e^{j\angle c}$$

is most convenient for multiplication and division, e.g.,

$$c_1 c_2 = |c_1| e^{j\angle c_1} |c_2| e^{j\angle c_2} = |c_1||c_2| e^{j(\angle c_1 + \angle c_2)}$$

The rectangular form for c, written

$$c = a + jb$$

is most convenient for addition and subtraction, e.g.,

$$c_1 + c_2 = a_1 + jb_1 + a_2 + jb_2 = (a_1 + a_2) + j(b_1 + b_2)$$

Of course, connections between the two forms of a complex number c include

$$|c| = |a + jb| = \sqrt{a^2 + b^2}\ , \quad \angle c = \angle(a + jb) = \tan^{-1}(b/a)$$

and, the other way round,

$$a = \mathrm{Re}\{c\} = |c|\cos(\angle c)\ , \quad b = \mathrm{Im}\{c\} = |c|\sin(\angle c)$$

Note especially that the quadrant ambiguity of the inverse tangent must be resolved in making these computations. For example,

$$\angle(1 - j) = \tan^{-1}(-1/1) = -\pi/4$$

while

$$\angle(-1 + j) = \tan^{-1}(1/(-1)) = 3\pi/4$$

It is important to be able to mentally compute the sine, cosine, and tangent of angles that are integer multiples of $\pi/4$, since many problems will be set up this way to avoid the distraction of calculators.

You should also be familiar with Euler's formula,

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

and the complex exponential representation for trigonometric functions:

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}\ , \quad \sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Notions of complex numbers extend to notions of complex-valued functions (of a real variable) in the obvious way. For example, we can think of a complex-valued function of time, $x(t)$, in the rectangular form

$$x(t) = \mathrm{Re}\{x(t)\} + j\,\mathrm{Im}\{x(t)\}$$

In a simpler notation this can be written as

$$x(t) = x_R(t) + j\,x_I(t)$$

where $x_R(t)$ and $x_I(t)$ are real-valued functions of t.

Or we can consider polar form,

$$x(t) = |x(t)|\, e^{j\angle x(t)}$$

where $|x(t)|$ and $\angle x(t)$ are real-valued functions of t (with, of course, $|x(t)|$ nonnegative for all t). In terms of these forms, multiplication and addition of complex functions can be carried out in the obvious way, with polar form most convenient for multiplication and rectangular form most convenient for addition..

In all cases, signals we encounter are functions of the real variable t. That is, while signals that are complex-valued functions of t will arise as mathematical conveniences, we will not deal with functions of a complex variable until near the end of the course.