H09-08 Solutions

Problem 4

The LTI system is stable, and the frequency response function is

$$\begin{aligned} H(\omega) &= \int_{-\infty}^{\infty} e^{-4|t|}\, e^{-j\omega t}\, dt = \int_{-\infty}^{0} e^{(4-j\omega)t}\, dt + \int_{0}^{\infty} e^{-(4+j\omega)t}\, dt \\ &= \frac{1}{4-j\omega} + \frac{1}{4+j\omega} = \frac{8}{16+\omega^2} \end{aligned}$$

(a) For the input

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t-n)$$

We see that $T_o = 1$, $\omega_o = 2\pi$, and the FS coefficients are

$$x_k = \int_{-1/2}^{1/2} \delta(t) e^{-jk2\pi}\, dt = 1$$

Thus

$$x(t) = \sum_{k=-\infty}^{\infty} x_k e^{jk2\pi t} = \sum_{k=-\infty}^{\infty} e^{jk2\pi t}$$

and the eigenfunction property gives

$$y(t) = \sum_{k=-\infty}^{\infty} x_k H(k2\pi) e^{jk2\pi t} = \sum_{k=-\infty}^{\infty} \frac{8}{16+(k2\pi)^2} e^{jk2\pi t}$$

We recognize this as a F.S. expression with coefficients

$$y_k = \frac{8}{16+4k^2\pi^2}$$

and $\omega_o = 2\pi$.

For the LTI system with $h(t) = e^{-4|t|}$, the frequency response function is

$$\begin{aligned} H(\omega) &= \int_{-\infty}^{\infty} e^{-4|t|}\, e^{-j\omega t}\, dt \\ &= \int_{-\infty}^{0} e^{(4-j\omega)t}\, dt + \int_{0}^{\infty} e^{-(4+j\omega)t}\, dt \\ &= \frac{1}{4-j\omega} + \frac{1}{4+j\omega} = \frac{8}{16+\omega^2} \end{aligned}$$

(The system is stable, so there is no problem with this calculation).

(b) For the input $x(t) = \sum n = -\infty\infty(-1)^n \delta(t-n)$, we see that $T_o = 2$, $\omega_o = \pi$???", and the FS coefficients are

$$\begin{aligned} x_k &= \frac{1}{2}\int_{-1/2}^{3/2} [\delta(t) - \delta(t-1)]\, e^{-jk\pi t} \\ &= \frac{1}{2}[1-(-1)^k] = \begin{cases} 0, & k \text{ even} \\ 1, & k \text{ odd} \end{cases} \end{aligned}$$

and

$$x(t) = \sum_{k=-\infty}^{\infty} x_k e^{jk\pi t}$$

By LTI and stability, the eigenfunction property gives

$$y(t) = \sum_{k=-\infty}^{\infty} x_k \; H(k\pi) \; e^{jk\pi t}$$

This is a FS expression with coefficients

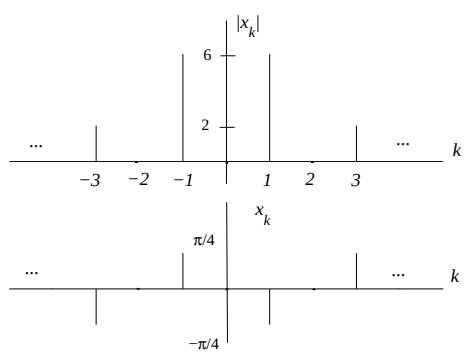
$$y_k = \begin{cases} 0, & k \; even \\ \frac{8}{16+k^2\pi^2}, & k \; odd \end{cases}$$

The system is a lowpass filter since y_k is small for $\mid k \mid$ large, i.e. $\mid H(\omega) \mid$ is small for ω large.

Problem 5

$$|x_1| = |\frac{6}{j} \; e^{j\frac{\pi}{4}}| = 6, \qquad |x_{-1}| = |x_1|$$
$$\angle x_1 = \angle(\frac{6}{j} \; e^{j\frac{\pi}{4}}) = -\frac{\pi}{4}, \qquad \angle x_{-1} = -\angle x_1$$
$$|x_3| = |\frac{6}{j3} \; e^{j\frac{3\pi}{4}}| = 2, \qquad |x_{-3}| = |x_3|$$
$$\angle x_3 = \frac{\pi}{4}, \qquad \angle x_{-3} = -\angle x_3$$

For all other k's, magnitudes and angles are zero



Problem 6

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} x_k \; e^{jk\omega_o t} \\ \Rightarrow y(t) &= \sum_{k=-\infty}^{\infty} y_k \; e^{jk\omega_o t} =_{k=-\infty}^{\infty} H(k\omega_o) x_k \; e^{jk\omega_o t} \end{aligned}$$

Therefore, $|y_k < |x_k|$ iff $|H(k\omega_o)| < 1$

(a) $\omega_o = 1, \quad \Rightarrow \quad |H(k\omega_o)| = |H(k)| = \frac{5}{sqrt9+k^2}$.
Thus $|y_k < |x_k|$ for all —k—¿4.

(b) $\omega_o = 2, \quad \Rightarrow \quad |H(k\omega_o)| = |H(2k)| = \frac{5}{sqrt9+4k^2}$.
Thus $|y_k < |x_k|$ for all —k—¿2.

Problem 7

$$h(t) = 25\ t\ e^{-3t}\ u(t)$$

This represents a stable LTI system with frequency response function.

$$\begin{aligned} H(\omega) &= \int_{-\infty}^{\infty} 25; t\ e^{-3t}\ u(t) e^{-j\omega t}\ dt \qquad \textit{Integration by part} \\ &= \frac{25}{(3+j\omega)^2} \end{aligned}$$

Note

$$|H(\omega)| = \frac{25}{9+\omega^2} \quad < 1 \quad for \quad \omega > 4$$

(a) $T_o = 2 \quad \Rightarrow \quad \omega_o = \pi \quad \Rightarrow \quad |y_k| < |x_k| \quad |k| > 1$

(b) $T_o = 4\pi \quad \Rightarrow \quad \omega_o = \frac{1}{2} \quad \Rightarrow \quad |y_k| < |x_k| \quad |k| > 8.$

Problem 8

$$\begin{aligned} h(t) &= \delta(t) - e^{-2t} u(t) \\ \Rightarrow H(\omega) &= \int_{-\infty}^{\infty} e^{-j\omega t}\ dt = \frac{1-j\omega}{2+j\omega} = \frac{\sqrt{1+\omega^2}}{\sqrt{4+\omega^2}} e^{j\angle\frac{1-j\omega}{2+j\omega}} \\ x(t) &= 1 + 2\cos(t) + 3\cos(2t) \\ &= \mathcal{R}]\{e^{j0t}\} + 2\mathcal{R}]\{e^{jt}\} + 3\mathcal{R}]\{e^{j2t}\} \end{aligned}$$

Eigenfunction property can be used, since the system is stable, to write

$$\begin{aligned} y(t) &= \mathcal{R}e\{H(0)\ e^{j0t}\} + 2\mathcal{R}e\{H(1)\ e^{jt}\} + 3\mathcal{R}e\{H(2)\ e^{j2t}\} \\ &= \frac{1}{2} + \frac{2\sqrt{2}}{\sqrt{5}} \cos(t + \angle\frac{1-j}{2+j}) + \frac{3\sqrt{5}}{2\sqrt{2}} \cos(2t + \angle\frac{1-j2}{2+j2}) \end{aligned}$$

Problem 9

$$h(t) = te^{-t}u(t) + 2e^{-t}u(t) - \delta(t)$$

Note the system is stable.

$$\begin{aligned} H(\omega) &= \int_{-\infty}^{\infty} h(t) e^{-j\omega t}\ dt = \int_0^{\infty} te^{-(1+j\omega)t} + 2\int_0^{\infty} e^{-(1+j\omega)t}\ dt - \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t}\ dt \\ &= \frac{1}{(1+j\omega)^2} + \frac{2}{1+j\omega} - 1 = \frac{2+\omega^2}{(1+j\omega)^2} \end{aligned}$$

Eigenfunction property:

$$
\begin{aligned}
x(t) &= 2e^{j0t} + 2\cos(t) = 2e^{j0t} + e^{jt} + e^{-jt} \\
\Rightarrow \quad y(t) &= 2H(0)e^{j0t} + H(1)e^{jt} + H(-1)e^{-jt} \\
H(0) &= 2, \quad H(1) = \frac{2}{3}e^{-j\frac{\pi}{2}}, \quad H(-1) = H^*(1) \\
\Rightarrow y(t) &= 4 + \frac{3}{2}e^{j(t-\frac{\pi}{2})} + \frac{3}{2}e^{-j(t-\frac{\pi}{2})} = 4 + 3\cos(t)
\end{aligned}
$$