H8-08 Solutions

4.

Three of the calculations are in an example in section 7.2

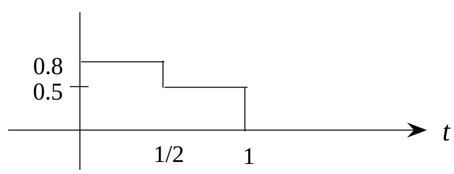
$$a_o = 1 - e^{-1} \approx 0.632$$
$$a_1 = 1 - 2e^{-1/2} + e^{-1} \approx 0.155$$
$$a_2 = 1 - 2e^{-1/4} + 2e^{-3/4} - e^{-1} \approx 0.019$$

Now,

$$a_3 = \int_0^{1/8} e^{-t}\,dt - \int_{1/8}^{3/8} e^{-t}\,dt + \int_{3/8}^{5/8} e^{-t}\,dt - \int_{5/8}^{7/8} e^{-t}\,dt + \int_{7/8}^{1} e^{-t}\,dt \approx 0.005$$

(a) The minimum ISE representation using the basis set $\phi_o(t)$, $\phi_1(t)$ is

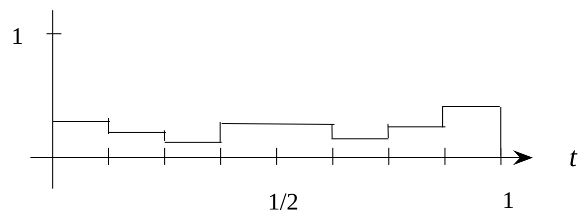
$$e^{-t} \approx 0.632\phi_o(t) + 0.155\phi_1(t), \qquad 0 \le t \le 1$$



(Looks reasonable!)

(b) The minimum ISE representation using the basis set $\phi_2(t)$, $\phi_3(t)$ is

$$e^{-t} \approx 0.0192\phi_2(t) + 0.005\phi_3(t), \qquad 0 \le t \le 1$$



(This looks pretty strange, but it is the best that $\phi_2(t)$, $\phi_3(t)$ can do in terms of ISE.)

(c) The minimum ISE representation using all 4 basis signals is

$$e^{-t} \approx 0.632\phi_o(t) + 0.155\phi_1(t) + 0.0192\phi_2(t) + 0.005\phi_3(t), \qquad 0 \le t \le 1$$

A plot of this looks somewhat reasonable.
Note: The point is that a presentation can be minimum ISE, but look rather poor if the basis set is not well chosen.

5.

For orthogonality, $\phi_o(t)$ and $\phi_1(t)$ must satisfy

$$1 = \int_0^\infty \phi_0^2(t)\, dt = \int_0^\infty a^2\, e^{-2t}\, dt = \frac{1}{2}a^2 = 2$$

Choose $a = \sqrt{2}$

$$\begin{aligned} 0 &= \int_0^\infty \phi_o(t)\phi_1(t)\,dt = \int_0^\infty ab\,e^{-2t} + ac\,e^{-3t}\,dt \\ &= \frac{\sqrt{2}}{2}b + \frac{\sqrt{2}}{3}c \end{aligned}$$

$$\Rightarrow \qquad b = -\frac{2}{3}c$$

$$\begin{aligned} 1 &= \int_0^\infty \phi_1^2(t)\,dt = \int_0^\infty b^2\,e^{-2t} + 2bc\,e^{-3t} + c^2\,e^{-4t}\,dt = \frac{1}{2}b^2 + \frac{2}{3}bc + \frac{1}{4}c^2 \\ &= \frac{4}{18}c^2 + \frac{4}{9}c^2 + \frac{1}{4}c^2 \\ \Rightarrow \qquad c^2 &= 36 \end{aligned}$$

Choose $c = 6 \quad \Rightarrow \quad b = -4$
(Other sign choices give other correct answers.)

2.

(a) If $x(t) = -x(-t)$, then

$$\begin{aligned} x_k &= \frac{1}{T_o}\int_{-T_o/2}^{T_o/2} x(t)e^{-jk\omega_o t}\,dt \\ &= -\frac{1}{T_o}\int_{-T_o/2}^{T_o/2} x(-t)e^{-jk\omega_o t}\,dt \end{aligned}$$

Let $\tau = -t$

$$\begin{aligned} x_k &= -\frac{1}{T_o}\int_{-T_o/2}^{T_o/2} x(\tau)e^{-jk\omega_o(-\tau)}\,(-d\tau) \\ &= -\frac{1}{T_o}\int_{-T_o/2}^{T_o/2} x(\tau)e^{-j(-k)\omega_o\tau}\,d\tau \\ &= -x_{-k} \end{aligned}$$

(This implies that $x_o = 0$. Also, since $x_{-k} = x_k^*$ this implies the x_k's have zero real part.)

(b) If $x(t) = -x(t + \frac{T_o}{2})$, then

$$\begin{aligned} x_k &= \frac{1}{T_o}\int_0^{T_o} x(t)e^{-jk\omega_o t}\,dt \\ &= \frac{1}{T_o}\int_0^{T_o/2} x(t)e^{-jk\omega_o t}\,dt + \frac{1}{T_o}\int_0^{T_o/2} x(t)e^{-jk\omega_o t}\,dt \end{aligned}$$

The first term can be written as

$$\frac{1}{T_o}\int_0^{T_o} x(t)e^{-jk\omega_o t}\,dt = \frac{1}{T_o}\int_0^{T_o/2} -x(t+\frac{T_o}{2})e^{-jk\omega_o t}\,dt + \frac{1}{T_o}\int_{T_o/2}^{T_o} x(t)e^{-jk\omega_o t}\,dt$$

Let $\tau = t + \frac{T_o}{2}$

$$\begin{aligned} &= \frac{1}{T_o}\int_{T_o/2}^{T_o} -x(\tau)e^{-jk\omega_o(\tau - T_o/2)}\,d\tau \\ &= \frac{1}{T_o}\int_{T_o/2}^{T_o} x(\tau)e^{-jk\omega_o\tau}\,d\tau\,(-e^{jk\pi}) \end{aligned}$$

If k is even, $-e^j k\pi = -1$ and this cancels the second term to give $x_k = 0$ for k even.

(c) If $x(t) = x(-t)$, then

$$\begin{aligned} x_k &= \frac{1}{T_o}\int_{-T_o/2}^{T_o/2} x(t)e^{-jk\omega_o t}\,dt \\ &= -\frac{1}{T_o}\int_{-T_o/2}^{T_o/2} x(-t)e^{-jk\omega_o t}\,dt \end{aligned}$$

Let $\tau = -t$

$$\begin{aligned} x_k &= \frac{1}{T_o}\int_{T_o/2}^{-T_o/2} x(\tau)e^{-jk\omega_o(-\tau)}\,(-d\tau) \\ &= -\frac{1}{T_o}\int_{-T_o/2}^{T_o/2} x(\tau)e^{-j(-k)\omega_o\tau}\,d\tau \\ &= x_{-k} \end{aligned}$$

3.

(a) The fundamental period for $\hat{x}(t)$ is also T_o.

$$\begin{aligned} \hat{x}_o &= \frac{1}{T_o}\int_{T_o} \hat{x}(t)\,dt = \frac{1}{T_o}\int_{T_o} 2x(t-3)+1\,dt \\ &= 2\cdot\frac{1}{T_o}\int_{T_o} x(t-3)\,dt + \frac{1}{T_o}\int_{T_o} 1\,dt = 2x_o + 1 \end{aligned}$$

For $k \neq 0$,

$$\begin{aligned} \hat{x}_k &= \frac{1}{T_o}\int_0^{T_o} \hat{x}(t)e^{-jk\omega_o t}\,dt \\ &= 2\frac{1}{T_o}\int_0^{T_o} x(t-3)e^{-jk\omega_o t}\,dt + \underbrace{\frac{1}{T_o}\int_0^{T_o} e^{-jk\omega_o t}\,dt}_{=0} \\ &= 2e^{-jk\omega_o 3}\frac{1}{T_o}\int_{-3}^{T_o-3} x(\tau)e^{-jk\omega_o\tau}\,d\tau = 2e^{-jk\omega_o 3}x_k \end{aligned}$$

(b) Flip and shift does not change the fundamental period, so $\hat{T}_o = T_0$, $\hat{\omega}_o = \omega_o$.

$$\begin{aligned} \hat{x}_k &= \frac{1}{\hat{T}_o}\int_0^{\hat{T}_o} \hat{x}(t)e^{-jk\hat{\omega}_o t}\, dt \\ &= \frac{1}{T_o}\int_0^{T_o} x(1-t)e^{-jk\omega_o t}\, dt \end{aligned}$$

Let $\tau = 1 - t$

$$\begin{aligned} \hat{x}_k &= \frac{1}{T_o}\int_1^{1-T_o} x(\tau)e^{-j(-k)\omega_o \tau}\, d\tau \\ &= e^{-jk\omega_o} x_{-k} \end{aligned}$$

(c) The derivative signal will have the same fundamental period.

$$\begin{aligned} \hat{x}_k &= \frac{1}{\hat{T}_o}\int_0^{\hat{T}_o} \hat{x}(t)e^{-jk\hat{\omega}_o t}\, dt = \frac{1}{T_o}\int_0^{T_o} \dot{x}(t)e^{-jk\omega_o t}\, dt \\ &= \frac{1}{T_o} x(t)e^{-jk\omega_o t}\, |_0^{T_o} + \frac{jk\omega_o}{T_o}\int_0^{T_o} x(t)e^{-jk\omega_o t}\, dt \end{aligned}$$

By integration-by-parts.
Now, the first term gives zero since both factors are T_o-periodic. Thus

$$\hat{x}_k = (jk\omega_o)\frac{1}{T_o}\int_0^{T_o} x(t)e^{-jk\omega_o}\, dt = (jk\omega_o)x_k$$

(d) If $x(t)$ has a nonzero DC term, then $\hat{x}(t) = \int_{-\infty}^t x(\sigma)\, d\sigma$ blows up. so we assume that

$$\int_{T_o} x(t)dt = 0$$

Then $\hat{x}(t)$ is periodic with period $\hat{T}_o = T_o$ and $\hat{\omega}_o = \omega_o$. Following the hint:
For $k \neq 0$

$$\hat{x}_k = \frac{1}{T_o}\int_0^{T_o} \hat{x}(t)e^{-jk\hat{\omega}_o t}\, dt = \frac{1}{T_o}\int_0^{T_o}\int_{-\infty}^t x(\sigma)\, d\sigma e^{-jk\omega_o t}\, dt$$

Let,

$$\begin{aligned} v &= \int x, \quad du = e^{-jk\omega_o t} \\ and \quad dv &= x, \quad u = \frac{e^{-jk\omega_o t}}{-jk\omega_o} \end{aligned}$$

$$\begin{aligned}
\hat{x}_k &= \frac{1}{T_o}\int_{-\infty}^{t} x(\sigma)\, d\sigma (\frac{-1}{jk\omega_o}) e^{-jk\omega_o t} \,|_0^{T_o} + \frac{1}{jk\omega_o} \cdot \frac{1}{T_o}\int_0^{T_o} x(t) e^{-jk\omega_o t}\, dt \\
&= \frac{1}{T_o}\int_{-\infty}^{T_o} x(\sigma)\, d\sigma (\frac{-1}{jk\omega_o}) \underbrace{e^{-jk\omega_o T_o}}_{=1} - \frac{1}{T_o}\int_{-\infty}^{0} x(\sigma)\, d\sigma (\frac{-1}{jk\omega_o}) \underbrace{e^{-jk\omega_o 0}}_{=1} + \frac{1}{jk\omega_o} x_k \\
&= \frac{1}{jk\omega_o} x_k
\end{aligned}$$

For $k = 0$, a similar calculation shows that $\hat{x}_o = 0$. Thus

$$\hat{x}_k = \begin{cases} 0, & k = 0 \\ \frac{1}{jk\omega_o} x_k, & k \neq 0 \end{cases}$$

Assuming $x_o = 0$

(e) $\hat{x}(t) = x(t + \frac{T_o}{2})$
(Shifted signal has same T_o, ω_o.)
FS coefficients for $\hat{x}(t)$ are

$$\begin{aligned}
\hat{x}_k &= \frac{1}{\hat{T}_o}\int_0^{\hat{T}_o} \hat{x}(t) e^{-jk\hat{\omega}_o t}\, dt \\
&= \frac{1}{T_o}\int_0^{T_o} x(t + \frac{T_o}{2}) e^{-jk\omega_o}\, dt
\end{aligned}$$

let $\tau = t + \frac{T_o}{2}$

$$\begin{aligned}
\hat{x}_k &= \frac{1}{T_o}\int_{T_o/2}^{3T_o/2} x(\tau) e^{-jk\omega_o(\tau - \frac{T_o}{2})}\, d\tau \\
&= e^{jk\omega_o \frac{T_o}{2}} x_k = e^{jk\pi} x_k = (-1)^k x_k
\end{aligned}$$