# H&-08A

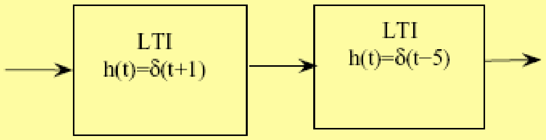
## Problem 7:

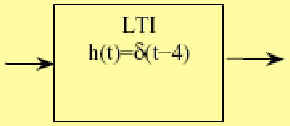
(a) False. For example $h(t) = u(t)$ gives

$$\int_{-\infty}^{\infty} \mid h(t) \mid dt = \int_{0}^{\infty} 1dt \rightarrow \infty$$

(b) True. For $\mid h(t) \mid$ will be periodic also, and will have positive area in one period.
Thus $\int_{-\infty}^{\infty} \mid h(t) \mid dt$ will be finite.

(c) False. For example a small time advance in cascade with a large decay is causal overall:





(d) True. A memoryless LTI system has unit impulse response of the form $h(t) = b\delta(t)$, where b is a constant. Thus,

$$\int_{-\infty}^{\infty} \mid h(t) \mid dt = \int_{-\infty}^{\infty} \mid b \mid \delta(t)dt = \mid b \mid < \infty$$

## Problem 8:

$$\int_{-\infty}^{t} e^{-2(t-\tau)} x(\tau - 1)d\tau$$

(a) Linearity is clear. For TI, let$\hat{x}(t) = x(t - t_o)$. Then

$$\begin{aligned}
\hat{y}(t) &= \int_{-\infty}^{t} e^{-2(t-\tau)} \hat{x}(\tau - 1) d\tau \\
&= \int_{-\infty}^{t} e^{-2(t-\tau)} x(\tau - 1 - t_o) d\tau
\end{aligned}$$

Let $\sigma = \tau - t_o$

$$\begin{aligned}
\hat{y}(t) &= \int_{-\infty}^{t-t_o} e^{-2(t-t_o-\sigma)} x(\sigma - 1) d\sigma \\
&= y(t - t_o)
\end{aligned}$$

(b) If $x(t) = \delta(t)$,

$$\begin{aligned}
h(t) &= \int_{-\infty}^{t} e^{-2(t-\tau)} \delta(\tau - 1) d\tau \\
&= \begin{cases} e^{-2(t-1)}, & t \geq 1 \\ 0, & t < 1 \end{cases} \\
&= e^{-2(t-1)} u(t-1)
\end{aligned}$$

(c) <u>First method:</u> Graphical convolution

$$\int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau = \int_{0}^{\infty} 1 dt \to \infty$$

For $t < 1$, $y(t) = 0$ (No overlap)
For $1 \leq t \leq 2$,

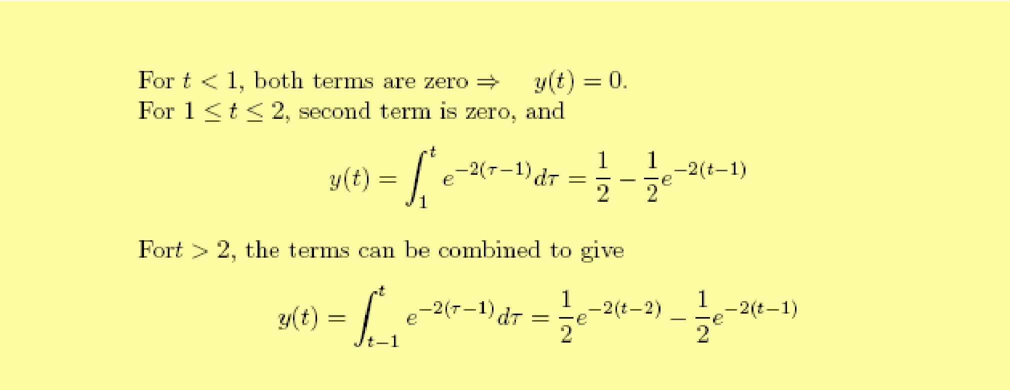
$$\begin{aligned}
y(t) &= \int_{1}^{t} e^{-2(\tau-1)} d\tau = e^2 \int_{1}^{t} e^{-2\tau} dt \\
&= \frac{1}{2} - \frac{1}{2} e^{-2(t-1)}
\end{aligned}$$

Fort $> 2$,

$$\begin{aligned}
y(t) &= \int_{t-1}^{t} e^{-2(\tau-1)} d\tau = e^2 \int_{t-1}^{t} e^{-2\tau} dt \\
&= \frac{1}{2} e^{-2(t-2)} - \frac{1}{2} e^{-2(t-1)}
\end{aligned}$$

<u>Second method:</u> Since the unit-step response is the running integral of $h(t)$

$$\begin{aligned}
y(t) &= \int_{-\infty}^{t} h(\tau) d\tau - \int_{-\infty}^{t-1} h(\tau) d\tau \\
&= \int_{-\infty}^{t} e^{-2(\tau-1)} u(\tau - 1) d\tau - \int_{-\infty}^{t-1} e^{-2(\tau-1)} u(\tau - 1) d\tau
\end{aligned}$$



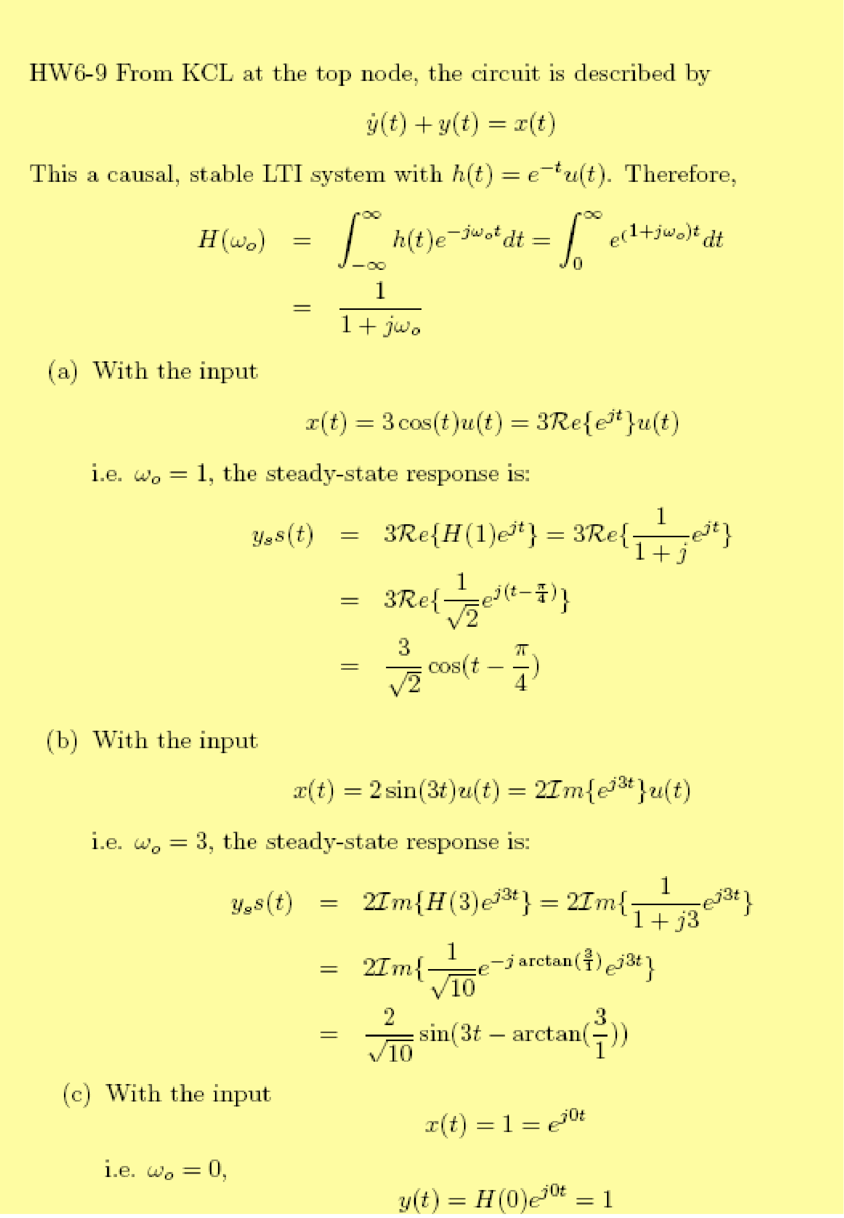
For $t < 1$, both terms are zero $\Rightarrow \quad y(t) = 0$.
For $1 \leq t \leq 2$, second term is zero, and

$$y(t) = \int_1^t e^{-2(\tau-1)} d\tau = \frac{1}{2} - \frac{1}{2} e^{-2(t-1)}$$

Fort $> 2$, the terms can be combined to give

$$y(t) = \int_{t-1}^t e^{-2(\tau-1)} d\tau = \frac{1}{2} e^{-2(t-2)} - \frac{1}{2} e^{-2(t-1)}$$

## Problem 9:



HW6-9 From KCL at the top node, the circuit is described by

$$\dot{y}(t) + y(t) = x(t)$$

This a causal, stable LTI system with $h(t) = e^{-t}u(t)$. Therefore,

$$\begin{aligned} H(\omega_o) &= \int_{-\infty}^{\infty} h(t) e^{-j\omega_o t} dt = \int_0^{\infty} e^{(1+j\omega_o)t} dt \\ &= \frac{1}{1 + j\omega_o} \end{aligned}$$

(a) With the input

$$x(t) = 3\cos(t)u(t) = 3\mathcal{R}e\{e^{jt}\}u(t)$$

i.e. $\omega_o = 1$, the steady-state response is:

$$\begin{aligned} y_s s(t) &= 3\mathcal{R}e\{H(1)e^{jt}\} = 3\mathcal{R}e\{\frac{1}{1+j} e^{jt}\} \\ &= 3\mathcal{R}e\{\frac{1}{\sqrt{2}} e^{j(t-\frac{\pi}{4})}\} \\ &= \frac{3}{\sqrt{2}} \cos(t - \frac{\pi}{4}) \end{aligned}$$

(b) With the input

$$x(t) = 2\sin(3t)u(t) = 2\mathcal{I}m\{e^{j3t}\}u(t)$$

i.e. $\omega_o = 3$, the steady-state response is:

$$\begin{aligned} y_s s(t) &= 2\mathcal{I}m\{H(3)e^{j3t}\} = 2\mathcal{I}m\{\frac{1}{1+j3} e^{j3t}\} \\ &= 2\mathcal{I}m\{\frac{1}{\sqrt{10}} e^{-j\arctan(\frac{3}{1})} e^{j3t}\} \\ &= \frac{2}{\sqrt{10}} \sin(3t - \arctan(\frac{3}{1})) \end{aligned}$$

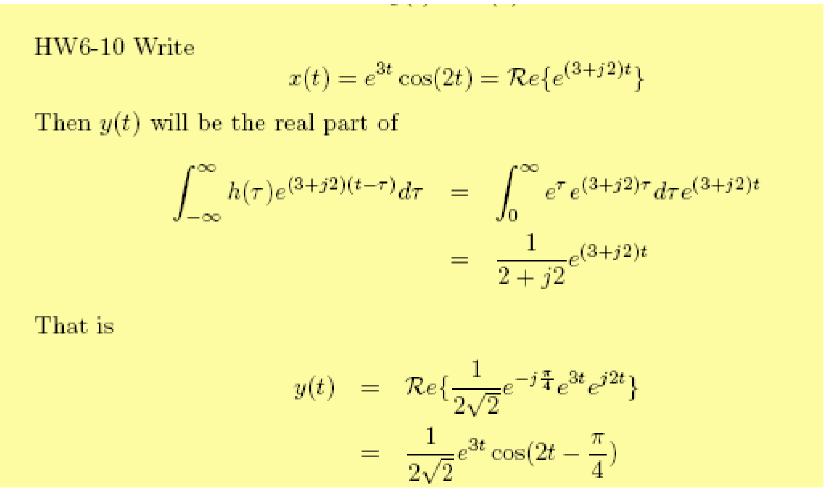
(c) With the input

$$x(t) = 1 = e^{j0t}$$

i.e. $\omega_o = 0$,

$$y(t) = H(0)e^{j0t} = 1$$

## Problem 10:



HW6-10 Write

$$x(t) = e^{3t}\cos(2t) = \mathcal{R}e\{e^{(3+j2)t}\}$$

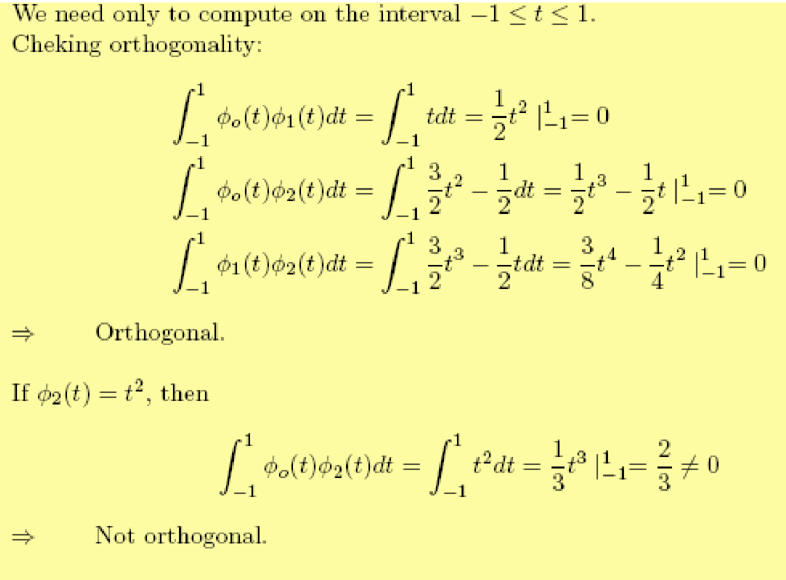
Then $y(t)$ will be the real part of

$$\begin{aligned} \int_{-\infty}^{\infty} h(\tau)e^{(3+j2)(t-\tau)}d\tau &= \int_0^{\infty} e^{\tau}e^{(3+j2)\tau}d\tau e^{(3+j2)t} \\ &= \frac{1}{2+j2}e^{(3+j2)t} \end{aligned}$$

That is

$$\begin{aligned} y(t) &= \mathcal{R}e\{\frac{1}{2\sqrt{2}}e^{-j\frac{\pi}{4}}e^{3t}e^{j2t}\} \\ &= \frac{1}{2\sqrt{2}}e^{3t}\cos(2t-\frac{\pi}{4}) \end{aligned}$$

## Problem 1:



We need only to compute on the interval $-1 \le t \le 1$.
Cheking orthogonality:

$$\int_{-1}^{1}\phi_o(t)\phi_1(t)dt = \int_{-1}^{1} tdt = \frac{1}{2}t^2 \mid_{-1}^{1} = 0$$

$$\int_{-1}^{1}\phi_o(t)\phi_2(t)dt = \int_{-1}^{1}\frac{3}{2}t^2 - \frac{1}{2}dt = \frac{1}{2}t^3 - \frac{1}{2}t \mid_{-1}^{1} = 0$$

$$\int_{-1}^{1}\phi_1(t)\phi_2(t)dt = \int_{-1}^{1}\frac{3}{2}t^3 - \frac{1}{2}tdt = \frac{3}{8}t^4 - \frac{1}{4}t^2 \mid_{-1}^{1} = 0$$

$\Rightarrow$ Orthogonal.

If $\phi_2(t) = t^2$, then

$$\int_{-1}^{1}\phi_o(t)\phi_2(t)dt = \int_{-1}^{1} t^2dt = \frac{1}{3}t^3 \mid_{-1}^{1} = \frac{2}{3} \neq 0$$

$\Rightarrow$ Not orthogonal.

## Problem 2:

(a) The product of an even signal and an odd signal is odd.

$$x_o(-t)x_e(-t) = -x_o(t)x_e(t)$$

Thus, for any $T > 0$

$$\int_{-T}^{T} x_o(t)x_e(t)dt = 0$$

(b) For $k \neq l$

$$0 = \int_{-1}^{1} \phi_k(t)\phi_l(t)dt$$

Let $\sigma = 3t$

$$\begin{aligned} \int_{-1}^{1} \phi_k(t)\phi_l(t)dt &= \int_{-3}^{3} \phi_k(\frac{\sigma}{3})\phi_l(\frac{\sigma}{3}) \cdot \frac{1}{3}d\sigma \\ \Rightarrow \qquad \int_{-3}^{3} \phi_k(\frac{t}{3})\phi_l(\frac{t}{3})dt &= 0, \qquad k \neq l \end{aligned}$$

$\Rightarrow$ The basis set $\psi_o(t), ..., \psi_{k-1}(t)$ is orthogonal on $-3 \leq t \leq 3$

(c) For $k \neq l$

$$0 = \int_{0}^{1} \phi_l(t)\phi_k(t)dt = \frac{1}{3}\int_{1/3}^{2/3} \phi_l(3\sigma - 1)\phi_k(3\sigma - 1)d\sigma$$

Let $\sigma = \frac{t+1}{3}$ i.e. $t = 3\sigma - 1$

$$\int_{0}^{1} \phi_l(t)\phi_k(t)dt = \frac{1}{3}\int_{1/3}^{2/3} \psi_l(\sigma)\psi_k(\sigma)d\sigma$$

$\Rightarrow$ , $\psi_k(t)$, $k = 0, ...K - 1$ is an orthogonal set on $\frac{1}{3} \leq t \leq \frac{2}{3}$

## Problem 3:

No, since

$$\begin{aligned} \int_{0}^{4} \phi_1(t)\phi_2(t)dt &= \int_{1}^{2} t\ dt + \int_{2}^{3} (2 - t)\ dt \\ &= 1 \neq 0 \end{aligned}$$