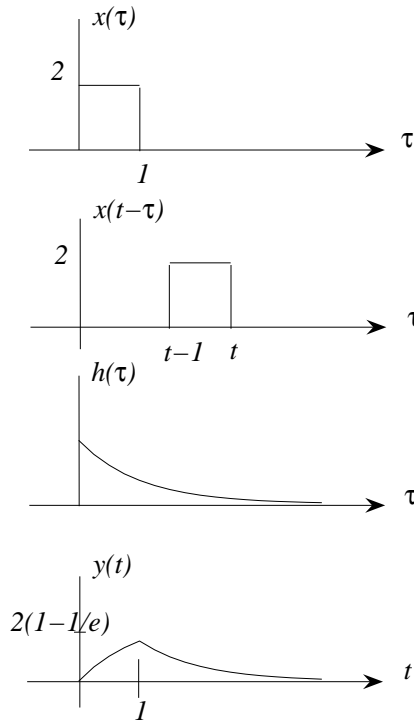


H6-08 Solutions

Problem 1a

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau$$



For $t < 0$, $y(t) = 0$ ("No overlap")

For $0 \leq t \leq 1$,

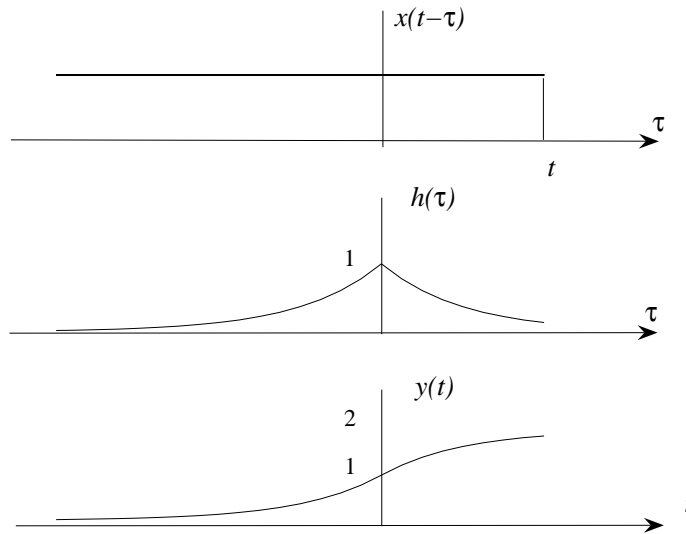
$$y(t) = \int_0^t 2e^{-\tau} d\tau = -2e^{-\tau} \Big|_0^t = 2 - 2e^{-t}$$

For $t > 1$,

$$\begin{aligned} y(t) &= \int_{t-1}^t 2e^{-\tau} d\tau - 2e^{-\tau} \Big|_{t-1}^t \\ &= -2e^{-t} + 2e^{-(t-1)} \\ &= 2(e-1)e^{-t} \end{aligned}$$

Problem 1b

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau$$



For $t \leq 0$,

$$y(t) = \int_{-\infty}^t e^{-|\tau|}d\tau = \int_{-\infty}^t e^{\tau}d\tau = e^t$$

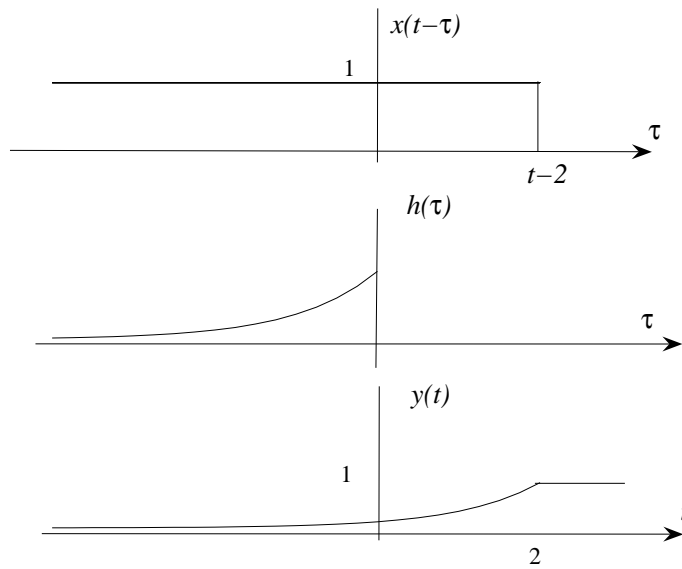
For $t > 0$,

$$\begin{aligned} y(t) &= \int_{-\infty}^t e^{-|\tau|}d\tau = \int_{-\infty}^0 e^{\tau}d\tau + \int_0^t e^{-\tau}d\tau \\ &= e^{\tau} \Big|_{-\infty}^0 - e^{-\tau} \Big|_0^t \\ &= 2 - e^{-t} \end{aligned}$$

Problem 1c

$$h(\tau) = e^{\tau}u(-\tau)$$

$$x(\tau) = u(\tau - 2)$$



For $t \leq 2$,

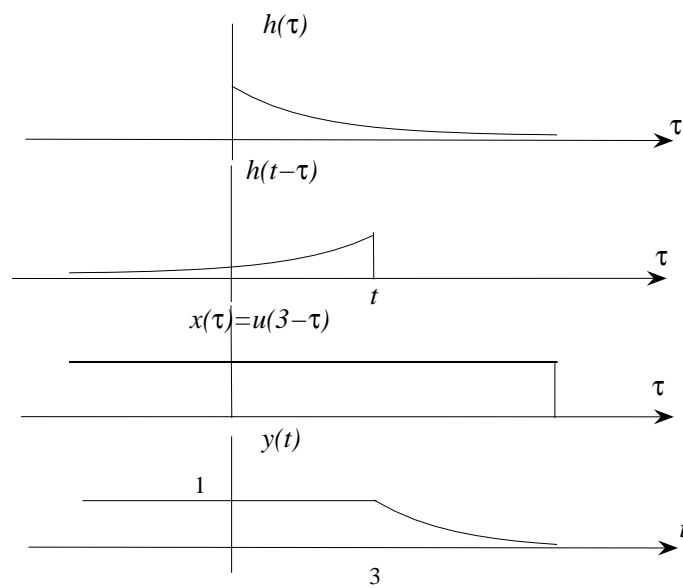
$$y(t) = \int_{-\infty}^{t-2} e^\tau \cdot 1 d\tau = e^{t-2}$$

For $t > 2$,

$$y(t) = \int_{-\infty}^0 e^\tau d\tau = 1$$

Problem 1d

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$



For $t \geq 3$,

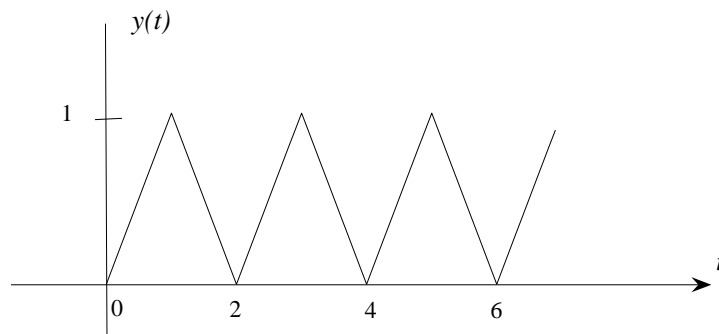
$$\begin{aligned} y(t) &= \int_{-\infty}^3 e^{(t-\tau)} d\tau = e^{-t} \int_{-\infty}^3 e^{(\tau)} d\tau \\ &= e^{-(t-3)} \end{aligned}$$

For $t < 3$,

$$\begin{aligned} y(t) &= \int_{-\infty}^t e^{-(t-\tau)} d\tau = e^{-t} \int_{-\infty}^t e^{(\tau)} d\tau \\ &= 1 \end{aligned}$$

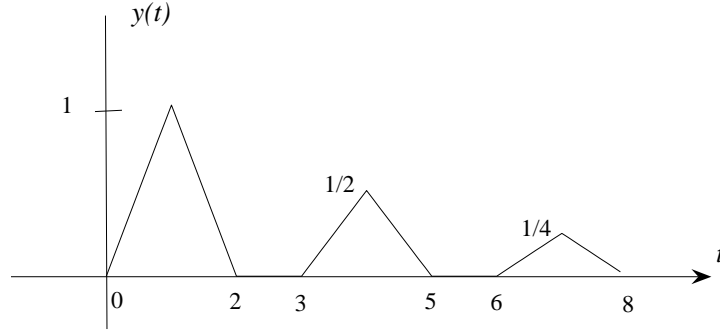
Problem 2b

$$y(t) = \sum_{k=0}^{\infty} h(t - 2k)$$



Problem 2c

$$y(t) = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k h(t - 3k)$$



Problem 4a

$$\hat{y}(t) = \int_{-\infty}^{\infty} \hat{x}(\tau) \hat{h}(t - \tau) d\tau = \int_{-\infty}^{\infty} x(\tau - 1) h(t - \tau - 2) d\tau$$

Let $\sigma = \tau - 1$

$$\hat{y}(t) = \int_{-\infty}^{\infty} x(\sigma) h(t - 3 - \sigma) d\sigma = y(t - 3)$$

Problem 4b

$$\hat{y}(t) = \int_{-\infty}^{\infty} \hat{x}(\tau) \hat{h}(t - \tau) d\tau = \int_{-\infty}^{\infty} x(\tau - 2) h(t - \tau + 2) d\tau$$

Let $\sigma = \tau - 2$

$$\hat{y}(t) = \int_{-\infty}^{\infty} x(\sigma) h(t - \sigma) d\sigma = y(t)$$

Problem 4e

$$\hat{y}(t) = \int_{-\infty}^{\infty} \hat{x}(\tau) \hat{h}(t - \tau) d\tau = \int_{-\infty}^{\infty} x(-\tau) h(-t + \tau) d\tau$$

Let $\sigma = -\tau$

$$\hat{y}(t) = \int_{-\infty}^{\infty} x(\sigma) h(-t + \sigma) d\sigma = y(-t)$$

Problem 6a

$$h(t) = e^{-2t}u(t+3)$$

Not causal, since e.g. $h(-1) \neq 0$

Stable, since

$$\begin{aligned} \int_{-\infty}^{\infty} |h(t)| dt &= \int_{-\infty}^{\infty} e^{-2t} dt \\ &= -\frac{1}{2}e^{-2t} \Big|_{-\infty}^{\infty} = \frac{1}{2}e^3 < \infty \end{aligned}$$

Problem 6b

$$h(t) = e^{3t}u(-4-t)$$

Not causal, since e.g. $h(-5) \neq 0$

Stable, since

$$\int_{-\infty}^{\infty} |h(t)| dt = \int_{-\infty}^{-4} e^{3t} dt = \frac{1}{3}e^{-12} < \infty$$

Problem 6c

$$h(t) = e^{-4|t|}$$

Not causal, since e.g. $h(-1) \neq 0$

Stable, since

$$\int_{-\infty}^{\infty} |h(t)| dt = \int_{-\infty}^{\infty} e^{-4|t|} dt = \int_{-\infty}^0 e^{4t} dt + \int_0^{\infty} e^{-4t} dt = \frac{1}{2} < \infty$$

Problem 6d

Causal, since $h(t)$ is right sided

Not stable, since

$$\int_{-\infty}^{\infty} |h(t)| dt = \int_3^{\infty} e^t dt \rightarrow \infty$$

Problem 8b

$$h[n] = \left(\frac{1}{2}\right)^n u[-n]$$

Not causal, since e.g. $h[-1] = 2$

Not stable, since

$$\begin{aligned} \sum_{n=-\infty}^{\infty} |h[n]| &= \sum_{n=-\infty}^0 \left(\frac{1}{2}\right)^n, \quad \text{let } k = -n \\ &= \sum_{k=0}^{\infty} 2^k \rightarrow \infty \end{aligned}$$

Problem 8c

$$h[n] = 2^n u[3 - n]$$

Not causal, since e.g. $h[-1] = \frac{1}{2}$

Stable, since

$$\begin{aligned} \sum_{n=-\infty}^{\infty} |h[n]| &= \sum_{n=-\infty}^3 2^n, \quad \text{let } k = -n \\ &= \sum_{k=-3}^{\infty} \frac{1}{2^k} \\ &= \left(\frac{1}{2}\right)^{-3} + \left(\frac{1}{2}\right)^{-2} + \left(\frac{1}{2}\right)^{-1} + \frac{1}{1 - \frac{1}{2}} < \infty \end{aligned}$$

Problem 9a

The system is causal and stable.

Write $x[n] = 1 = e^{\sigma_o n}$, with $\sigma_o = 0$

Then

$$\begin{aligned} y[n] &= H(0)e^{\sigma_o n}, \\ H(\sigma_o) &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-\sigma_o n}, \quad \text{and} \\ H(0) &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 2, \\ \Rightarrow \\ y[n] &= 2, \quad -\infty < n < \infty \end{aligned}$$

Note the system is causal and stable.

Problem 9b Write $x[n] = (-1)^n = e^{j\pi n}$, then

$$\begin{aligned} y[n] &= H(\pi)e^{j\pi n}, \\ H(\omega_o) &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-\omega_o n}, \quad \text{and} \\ H(\pi) &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-j\pi n}, \\ &= \sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3} \\ \Rightarrow \\ y[n] &= \frac{2}{3}e^{j\pi n} = (-1)^n \frac{2}{3}, \quad -\infty < n < \infty \end{aligned}$$

Note the system is causal and stable.

Problem 9c

$$\begin{aligned}
x[n] &= 2 \cos\left(\frac{\pi}{2}n\right) = 2\mathcal{R}e\{e^{j\frac{\pi}{2}n}\} \\
\Rightarrow y[n] &= 2\mathcal{R}e\{H\left(\frac{\pi}{2}\right)e^{j\frac{\pi}{2}n}\}, \\
H\left(\frac{\pi}{2}\right) &= \sum_{n=0}^{\infty} h[n]e^{-j\frac{\pi}{2}n} = \sum_{n=0}^{\infty} \left(\frac{-j}{2}\right)^n \\
&= \frac{1}{1 + \frac{j}{2}} = \frac{2}{2 + j} \\
\Rightarrow y[n] &= 2\mathcal{R}e\left\{\frac{2}{2 + j}e^{j\frac{\pi}{2}n}\right\} \\
&= 2\mathcal{R}e\left\{\frac{2}{\sqrt{5}}e^{j\left(\frac{\pi}{2}n - \angle(2+j)\right)}\right\} \\
&= \frac{4}{\sqrt{5}} \cos\left(\frac{\pi}{2}n - \angle(2 + j)\right)
\end{aligned}$$

Problem 9d

$$\begin{aligned}
x[n] &= 3 \sin\left(-\frac{3\pi}{2}n\right) = 3 \sin\left(\frac{\pi}{2}n\right) \\
&= 3\mathcal{I}m\{e^{j\frac{\pi}{2}n}\} \\
\Rightarrow y[n] &= 3\mathcal{I}m\{H\left(\frac{\pi}{2}\right)e^{j\frac{\pi}{2}n}\} \\
H\left(\frac{\pi}{2}\right) &= \sum_{n=0}^{\infty} h[n]e^{-j\frac{\pi}{2}n} \\
&= \sum_{n=0}^{\infty} \left(\frac{-j}{2}\right)^n = \frac{1}{1 + \frac{j}{2}} = \frac{2}{2 + j} \\
\Rightarrow y[n] &= 3\mathcal{I}m\left\{\frac{2}{2 + j}e^{j\frac{\pi}{2}n}\right\} \\
&= 3\mathcal{I}m\left\{\frac{2}{2 + j}e^{j\frac{\pi}{2}n}\right\} \\
&= 3\mathcal{I}m\left\{\frac{2}{\sqrt{5}}e^{j\left(\frac{\pi}{2}n - \angle(2+j)\right)}\right\} \\
&= \frac{6}{\sqrt{5}} \sin\left(\frac{\pi}{2}n - \angle(2 + j)\right)
\end{aligned}$$