H4-08 Solutions

Problem 1:
For any n

$$\begin{aligned} x[n+N] &= c_1 e^{j\omega_1(n+N)} + c_2 e^{j\omega_2(n+N)} \\ &= c_1 e^{j\omega_1 n} e^{j\omega_1 k_1 N_1} + c_2 e^{j\omega_2 n} e^{j\omega_2 k_2 N_2} \\ &= c_1 e^{j\omega_1 n} e^{j m_1 k_1 2\pi} + c_2 e^{j\omega_2 n} e^{j m_2 k_2 2\pi} \\ &= c_1 e^{j\omega_1 n} + c_2 e^{j\omega_2 n} \\ &= x[n] \end{aligned}$$

$x[n]$ is N-periodic.

Problem 2(b)

$$y(t) = \cos^2(t) x(t)$$

<u>CAUSAL</u>: $y(t)$ depends only on $x(t)$.

<u>MEMORYLESS</u>: $y(t)$ depends on $x(t)$

<u>NOT TIME INVARIANT</u>: If $\hat{x}(t) = x(t - t_o)$, then

$$\begin{aligned} \hat{y}(t) &= \cos^2(t)\hat{x}(t) \\ &= \cos^2(t) x(t - t_o) \\ &\neq y(t - t_o) \end{aligned}$$

<u>LINEAR</u>: If $x(t) = x_1(t) + \alpha x_2(t)$, then

$$\begin{aligned} y(t) &= \cos^2(t) x(t) \\ &= \cos^2(t) x_1(t) + \alpha \cos^2(t) x_2(t) \\ &= y_1(t) + y_2(t) \end{aligned}$$

<u>STABLE</u>: If

$$\begin{aligned} \mid x(t) \mid &\leq K, \qquad \forall\, t,\ \textit{then} \\ \mid y(t) \mid &\leq K, \qquad \forall\, t. \end{aligned}$$

P.S: For TI, equality of $\hat{y}(t)$ and $y(t-t_o)$ must hold for every t_o, not just some t_o!

Problem 2(d)

CAUSAL: Easy.

NOT MEMORYLESS: Easy.

TIME INVARIANT: $\hat{x}(t) = x(t-t_o)$, yields

$$\begin{aligned} \hat{y}(t) &= e^{-t}\int_{-\infty}^{t} e^{\tau}\hat{x}(\tau)d\tau \\ &= e^{-t}\int_{-\infty}^{t} e^{\tau}\hat{x}(\tau - t_o)d\tau \end{aligned}$$

Also,

$$y(t-t_o) = e^{t-t_o}\int_{-\infty}^{t-t_o} e^{\tau}x(\tau)d\tau$$

Let $\sigma = \tau + t_o$

$$\begin{aligned} y(t-t_o) &= e^{t-t_o}\int_{-\infty}^{t} e^{\sigma - t_o}x(\sigma - t_o)d\sigma \\ &= e^{t}\int_{-\infty}^{t} e^{\sigma}x(\sigma - t_o)d\sigma \\ &= \hat{y}(t) \end{aligned}$$

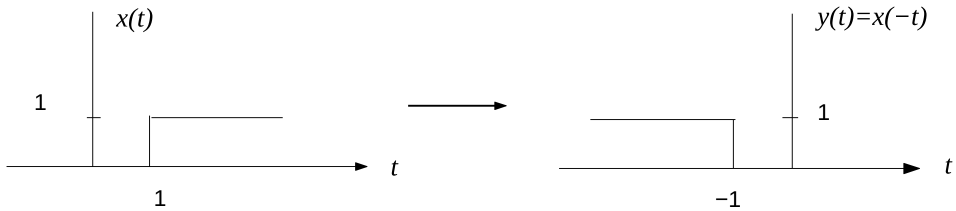
LINEAR: Easy

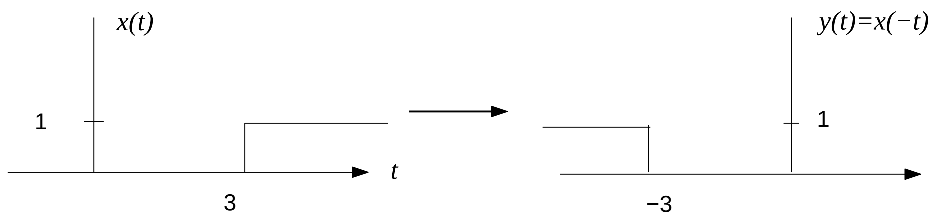
STABLE:

$$\begin{aligned} \mid x(t) \mid &\leq K, \qquad \forall\, t \\ \Rightarrow \mid y(t) \mid &= \mid e^{-t}\int_{-\infty}^{t} e^{\tau}x(\tau)d\tau \mid \\ &= e^{-t} \mid \int_{-\infty}^{t} e^{\tau}x(\tau)d\tau \mid \\ &\leq e^{-t}\int_{-\infty}^{t} \mid e^{\tau}x(\tau) \mid d\tau \\ &\leq e^{-t}\int_{-\infty}^{t} e^{\tau}K d\tau \\ &= K \end{aligned}$$

Problem 2(f)
$y(t) = x(-t)$

- NOT CAUSAL: At $t = -1$, $y(-1) = x(1)$
- NOT MEMORYLESS: $y(1) = x(-1)$.
- Not TIME INVARIANT: Use an example
  For $t_o = 2$





  Clearly $\hat{y}(t) \neq y(t-2)$

- LINEAR: If $x_1(t)$ yields $y_1(t)$ and $x_2(t)$ yields $y_2(t)$, then

$$\begin{aligned} \hat{x}(t) &= bx_1(t) + x_2(t) \qquad \textit{yields} \\ \hat{y}(t) &= \hat{x}(-t) = bx_1(-t) + x_2(-t) \\ &= by_1(t) + y_2(t) \end{aligned}$$

- STABLE:

$$\begin{aligned} \mid x(t) \mid &\leq K, \qquad \forall\, t,\ \textit{implies} \\ \mid y(t) \mid = \mid x(-t) \mid &\leq K, \qquad \forall\, t. \end{aligned}$$

Problem 2h
$y(t) = \int_{-\infty}^{t} e^{(t-\sigma)} x^2(\sigma) d\sigma$

<u>CAUSAL</u>: Since $y(t)$ depends only on the input values before t.

<u>NOT MEMORYLESS</u>

<u>NOT STABLE</u>: Since $x(t) = 1$, $-\infty < t < \infty$, yields

$$y(t) = \int_{-\infty}^{t} e^{(t-\sigma)} d\sigma = -e^{(t-\sigma)} \mid_{\sigma=-\infty}^{\sigma=t} \rightarrow \infty$$

<u>NOT LINEAR</u>: Since $\hat{x}(t) = 2x(t)$ yields $\hat{y}(t) = 4y(t)$

<u>TIME INVARIANT</u>: Since $\hat{x}(t) = x(t - t_o)$ yields

$$\begin{aligned}
\hat{y}(t) &= \int_{-\infty}^{t} e^{(t-\sigma)} x^2(\sigma) d\sigma \\
\hat{y}(t) &= \int_{-\infty}^{t} e^{(t-\sigma)} x^2(\sigma - t_o) d\sigma
\end{aligned}$$

Let $\tau = \sigma - t_o$

$$\begin{aligned}
\hat{y}(t) &= \int_{-\infty}^{t} e^{(t-t_o-\tau)} x^2(\tau) d\tau \\
&= y(t - t_o)
\end{aligned}$$

Problem 2j

$y(t) = 3x(t + 1) - 4$

- <u>N</u>OT CAUSAL
- <u>N</u>OT MEMORYLESS
- <u>T</u>IME-INVARIANT: $y(t - t_0) = x(t - t_0)$.
- <u>N</u>ON-LINEAR: Can see by direct substitution
- <u>S</u>TABLE: $y(t) < 3M - 4$ when $|x(t) < M$, any M.

Problem 2l

$y(t) = 3x(t) - |x(t - 3)|$

- <u>CAUSAL</u>

- NOT MEMORYLESS
- TIME-INVARIANT
- NONLINEAR because $|a+b| \neq |a| + |b|$.
- STABLE

Problem 3a

$y[n] = 3x[n]x[n-1]$

CAUSAL: $y[n]$ depends only on $x[n-1]$, $x[n]$

NOT MEMORYLESS: $y[n]$ depends on $x[n-1]$.

TIME INVARIANT: If $\hat{x}[n] = x[n-n_o]$, then

$$\begin{aligned}\hat{y}[n] &= 3\hat{x}[n]\hat{x}[n-1] \\ &= 3x[n-n_o]x[n-n_o-1] \\ &= y[n-n_o]\end{aligned}$$

NOT LINEAR: If $\hat{x}[n] = 2x[n]$, then

$$\begin{aligned}\hat{y}[n] &= 3\hat{x}[n]\hat{x}[n-1] \\ &= 12x[n]x[n-1] \\ &\neq 2y[n]\end{aligned}$$

STABLE: If $\mid x[n] \mid \leq K$, then $\mid y[n] \mid \leq 3K^2$, $\forall \quad n$

Problem 3c

$y[n] = 4x[3n-2]$

NOT CAUSAL: e.g. $y[2] = 4x[4]$.

NOT MEMORYLESS: Same as above.

NOT TIME INVARIANT: If $\hat{x}[n] = x[n-n_o]$, then

$$\begin{aligned}\hat{y}[n] &= 4\hat{x}[3n-2] \\ &= 4x[3n-2-n_o]\end{aligned}$$

But

$$\begin{aligned} y[n-n_o] &= 4x[3(n-n_o)-2] \\ &= 4x[3n-3n_o-2] \end{aligned}$$

For specific example, compare the responses to $\delta[n]$ and $\delta[n-1]$

<u>LINEAR</u>: Easy

<u>STABLE</u>:

$$\begin{aligned} |\,x[n]\,| &\leq K, \quad \forall \quad n \\ \Rightarrow |\,y[n]\,| &\leq 4K, \quad \forall \quad n \end{aligned}$$

Problem 3e

- CAUSAL
- NOT TIME-INVARIANT
- NOT MEMORYLESS
- NOT LINEAR
- STABLE

Problem 4a

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

Invertible. Noting that

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{n} x[k] \\ y[n-1] &= \sum_{k=-\infty}^{n-1} x[k] \end{aligned}$$

We see that the inverse system is

$$x[n] = y[n] - y[n-1]$$

Problem 4b

NOT INVERTIBLE. $y[1] = 0$ for any input.

Problem 4c

$$y[n] = x[n] - x[n-1]$$

Invertible with inverse system

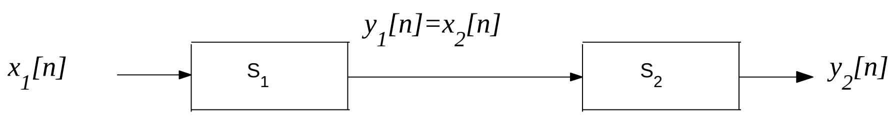
$$z[n] = \sum_{k=-\infty}^{n} y[k]$$

Since

$$\begin{aligned} z[n] &= y[n] + y[n-1] + y[n-2] + \cdots \\ &= (x[n] - x[n-1]) + (x[n-1] - x[n-2]) + \cdots \\ &= x[n] \end{aligned}$$

Problem 5a

$S_2(S_1)$ is the cascade connection.



$$\begin{aligned} y_1[n] &= x_1^2[n-2] = x_2[n] \\ y_2[n] &= 3x_2[n+2] = 3x_1^2[n+2-2] \\ &= 3x_1^2[n] \end{aligned}$$

Problem 5b:

$$
\begin{aligned}
y_1[n] &= \sum_{k=-\infty}^{n} \delta[k] x_1[n-k] \\
&= \sum_{\ell=0}^{\infty} \delta[n-\ell] x_1[\ell] \\
&= x_1[n]
\end{aligned}
$$

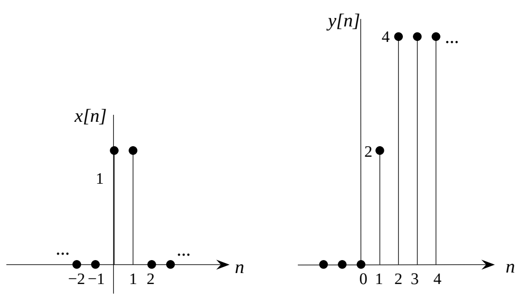
Similarly,

$$y_2[n] = 2x_2[n]$$

But $x_2[n] = y_1[n] = x_1[n]$. Therefore

$$y_2[n] = 2x_1[n]$$

Problem 6:

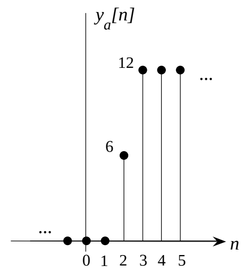


(a)

$$x_a[n] = 3x[n-1]$$

So LTI gives

$$y_a[n] = 3y[n-1]$$



(b) From sketch,

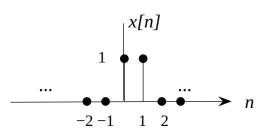
$$x_b[n] = x[n] - x[n-1]$$

i.e.

$$x_b[n] = u[n] - u[n-2] - [u[n-2] - u[n-3]]$$

By LTI

$$y_b[n] = y[n] - y[n-1]$$



(c) Write

$$\begin{aligned} x_c[n] &= u[n] - u[n-2] + [u[n-2] - u[n-4]] \\ &= x[n] + x[n-2] \end{aligned}$$

Then, by LTI,

$$\begin{aligned} y_c[n] &= y[n] + y[n-2] \\ &= 2r[n] - 2r[n-2] + 2r[n-2] - 2r[n-4] \\ &= 2r[n] - 2r[n-4] \end{aligned}$$