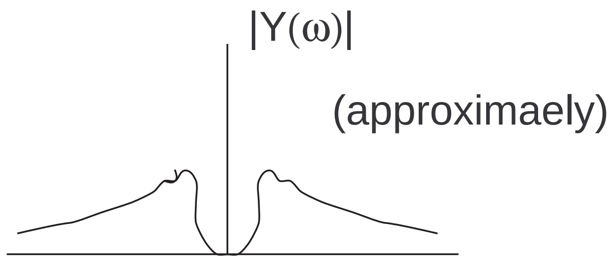
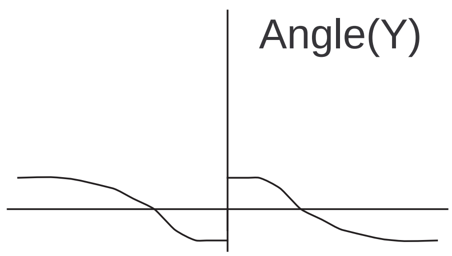
H12-08

Prob. 8

(a)

$$\begin{aligned}
Y(\omega) &= \frac{j\omega}{(2+j\omega)^2} \quad , By\ diff.\ property \\
|Y(\omega)| &= \frac{|\omega|}{4+\omega^2} \quad ; e.g. |Y(0)| = 0,\ |Y(2)| = \frac{1}{4},\ |Y(\infty)| = 0 \\
\angle Y(\omega) &= \frac{\pi}{2} - 2\tan^{-1}(\frac{\omega}{2}) \quad ; e.g. \angle Y(0) = \frac{\pi}{2},\ \angle Y(2) = 0,\ \angle Y(\infty) = -\frac{\pi}{2}
\end{aligned}$$





(b)

$$Y(\omega) = \frac{1}{(j\omega)(2+j\omega)^2} + \frac{\pi}{4}\delta(\omega)$$

Consider the two terms to be "non-overlapping".

$$|\frac{1}{(j\omega)(2+j\omega)^2}| = \frac{1}{|\omega|(4+\omega^2)} \quad ; e.g. |Y(0)| = \infty,\ |Y(2)| = \frac{1}{16},\ |Y(\infty)| = 0$$

$$\angle Y(\omega) = -\frac{\pi}{2} - 2\tan^{-1}(\frac{\omega}{2}) \quad ; e.g. \angle Y(0) = frac\pi 2,\ \angle Y(2) = -\pi,\ \angle Y(\infty) = -\frac{\pi}{2} - \pi = \frac{-3\pi}{2} = \frac{\pi}{2}$$

(c)

$$\begin{aligned}
Y(\omega) &= \int_{-\infty}^{\infty} x(-2t+4)e^{-j\omega t}dt \\
&= ... = \frac{1}{2}e^{-j2\omega}X(\frac{\omega}{-2}) = \frac{\frac{1}{2}e^{-j2\omega}}{(2+j\frac{\omega}{-2})^2} \\
&= \frac{2e^{-j2\omega}}{(4-j\omega)^2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
|Y(\omega)| &= \frac{2}{16+\omega^2} \\
\angle Y(\omega) &= -2\omega - 2\angle(4-j\omega) = -2\omega + 2\tan^{-1}(\frac{\omega}{4}) \\
e.g. \quad \angle Y(0) &= 0, \ \angle Y(4) = (-8 + \frac{(}{\pi})(2))rad, \ etc.
\end{aligned}$$

(d)

$$\begin{aligned}
Y(\omega) = 2X(\omega) + j\omega X(\omega) &= \frac{1}{2+j\omega} \\
|Y(\omega)| &= \frac{1}{\sqrt{2+\omega^2}} \\
\angle Y(\omega) &= -\tan^{-1}(\omega)
\end{aligned}$$

Prob. 9

HW10-9 Two LTI systems are specified by the unit-impulse responses $h_1(t) = -2\delta(t) + 5e^{-2t}u(t)$ and $h_2(t) = 2te^{-t}u(t)$. Compute the responses of the two systems to the input signal $x(t) = \cos(t)$.

$$\begin{aligned}
X(\omega) &= \pi\delta(\omega-1) + \pi\delta(\omega+1) \\
H_1(\omega) &= -2 + \frac{5}{2+j\omega} = \frac{1-2j\omega}{2+j\omega} \\
Y_1(\omega) &= H_1(\omega)X(\omega) = \frac{\pi(1-2j\omega)}{2+j\omega}\delta(\omega-1) + \frac{\pi(1-2j\omega)}{2+j\omega}\delta(\omega+1) \\
&= \frac{\pi(1-2j\omega)}{2+j\omega}\delta(\omega-1) + \frac{\pi(1+2j)}{2-j}\delta(\omega+1) \\
&= \pi e^{j(\tan^{-1}(-2) - tan^{-1}(\frac{1}{2}))}\delta(\omega-1) + \pi e^{j(\tan^{-1}(2) + tan^{-1}(\frac{1}{2}))}\delta(\omega+1) \\
&= -j\pi\delta(\omega-1) + j\pi\delta(\omega+1) \\
\Rightarrow \quad y_1(t) &= \sin(t) \\
H_2 &= \frac{2}{(1+j\omega)^2} \\
\Rightarrow \quad Y_2(\omega) &= H_2(\omega)X(\omega) = \frac{2\pi}{1+j}\delta(\omega-1) + \frac{2\pi}{1-j}\delta(\omega+1) \\
&= \frac{2\pi}{(\sqrt{2}e^{j\frac{\pi}{4}})^2}\delta(\omega-1) + \frac{2\pi}{(\sqrt{2}e^{-j\frac{\pi}{4}})^2}\delta(\omega+1) \\
&= \frac{\pi}{2}\delta(\omega-1) - \frac{\pi}{2}\delta(\omega+1) \\
\Rightarrow \quad y_2(t) &= \sin(t)
\end{aligned}$$

(Of course, different systems can have the same responses to a particular input signal.)

Prob. 10

$$y(t) = \delta(t) + 3e^{-t}u(t) - 7e^{-2t}u(t)$$

$$\Rightarrow \qquad Y(\omega) = 1 + \frac{3}{1+j\omega} - \frac{7}{2+j\omega} = \frac{(j\omega)^2 - j\omega + 1}{(1+j\omega)(2+j\omega)}$$

$$= H(\omega)X(\omega) = \frac{j\omega}{1+j\omega}X(\omega)$$

$$\Rightarrow \qquad X(\omega) = \frac{(j\omega)^2 - j\omega + 1}{j\omega(2+j\omega)} = 1 + \frac{(1-2j\omega)}{j\omega(2+j\omega)}$$

PFE on the second term leads to

$$X(\omega) = 1 + \frac{1}{j\omega} - \frac{3}{1+j\omega}$$

From the table, and from class,

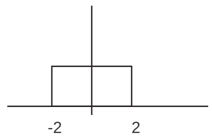
$$x(t) = \delta(t) + \frac{1}{2}sign(t) - 3e^{-t}u(t)$$

Prob 11

Suppose $y(t) = x(t)\cos(t)$ and the Fourier transform of $y(t)$ is described in terms of unit-step functions as

$$Y(\omega) = u(\omega + 2) - u(\omega - 2)$$

What is $x(t)$? Since $Y(\omega)$ is as shown,



it follows that $X(\omega) = 2, \ -1 \le \omega \le 1$, and 0 otherwise..

By frequency-domain convolution,

$$\begin{aligned} Y(\omega) &= f[x(t)\cos(t)] = \frac{1}{2\pi} X(\omega) * [\pi\delta(\omega - 1) + \pi\delta(\omega + 1)] \\ &= \frac{1}{2} X(\omega - 1) + \frac{1}{2} X(\omega - +1) \end{aligned}$$

Since $Y(\omega)$ is as shown, it follows that $X(\omega)$ is Then, the Fourier transform table gives

$$x(t) = \frac{2\sin(t)}{\pi t}$$

Prob. 15

$$X(\omega) = e^{-j\omega 3} \quad \Rightarrow \quad x(t) = \delta(t - 3)$$

$$H(\omega) = \frac{2}{2 - \omega^2 + j\omega 3} = \frac{2}{2 - \frac{(j\omega)^2}{j^2} + j\omega 3} = \frac{2}{(j\omega)^2 + 3j\omega + 2}$$

$$= \frac{2}{(1 + j\omega)(2 + j\omega)}$$

Let $\hat{x}(t) = \delta(t)$. Then $\hat{X}(\omega) = 1$ and the response is

$$\hat{Y}(\omega) = H(\omega)\hat{X}(\omega) = \frac{2}{(1 + j\omega)(2 + j\omega)} = \frac{2}{1 + j\omega} - \frac{2}{2 + j\omega}$$

$$\Rightarrow \quad \hat{y}(t) = 2e^{-t}u(t) - 2e^{-2t}u(t)$$

Using the TI property, the response to $x(t)$ is

$$y(t) = \hat{y}(t - 3) = 2e^{-(t-3)}u(t - 3) - 2e^{-2(t-3)}u(t - 3)$$

Prob. 16

Use the partial fraction expansion to compute the inverse Fourier transform for

(a) $X(\omega) = \frac{5j\omega + 12}{(j\omega)^2 + 5j\omega + 6}$

(b) $X(\omega) = \frac{4}{(3 - \omega^2 + 4j\omega}$

(a) Substitute v for $j\omega$:

$$\frac{5v + 12}{v^2 + 5v + 6} = \frac{5v + 12}{(v + 2)(v + 3)} = \frac{2}{v + 2} + \frac{3}{v + 3}$$

$$\Rightarrow \quad X(\omega) = \frac{2}{2+j\omega} + \frac{3}{3+j\omega}$$
$$\Rightarrow \quad x(t) = 2e^{-2t}u(t) + 3e^{-3t}u(t)$$

(b) Substitute $v = j\omega$ i.e.$\omega = \frac{v}{j}$ to write:

$$\frac{4}{(3-(\frac{v}{j})^2+4v} = \frac{4}{v^2+4v+3} = \frac{4}{(v+1)(v+3)} = \frac{2}{v+1} - \frac{2}{v+3}$$
$$\Rightarrow \quad X(\omega) = \frac{2}{1+j\omega} - \frac{2}{3+j\omega}$$
$$\Rightarrow \quad x(t) = 2e^{-t}u(t) - 2e^{-3t}u(t)$$

Prob. 17

(a) Since $e^{j(\pi-\pi\omega)} = -e^{-j\omega\pi}$, we compute the inverse transform for the rational part, and then use the delay property. Replacing ω by $\frac{j\omega}{j}$ gives

$$\frac{-5+\omega^2-j4\omega}{(9-\omega^2+j6\omega)(2+j\omega)} = \frac{-5+\frac{(j\omega)^2}{j^2}-j4\frac{j\omega}{j}}{(9-\frac{(j\omega)^2}{j^2}+j6\frac{j\omega}{j})(2+j\frac{j\omega}{j})}$$

Thus the rational part yields

$-e^{-2t}u(t) + 2te^{-3t}u(t)$

and then the delay property gives

$x(t) = -e^{-2(t-\pi)}u(t-\pi) + 2(t-\pi)e^{-3(t-\pi)}u(t-\pi)$

(b)

$f^{-1}[e^{-j\omega 2}\cdot 1] = \delta(t-2)$, From table and shift property

$$\frac{10}{2-\omega^2+j3\omega} = \frac{10}{(j\omega)^2+3(j\omega)+2}, \quad \text{by writing} \omega^k = \frac{(j\omega)^k}{j^k}$$
$$= \frac{10}{(1+j\omega)(2+j\omega)}$$

Partial fraction expansion gives

$$= \frac{10}{1+j\omega} - \frac{10}{2+j\omega}$$

Using the shift property again, and the table,

$$f^{-1}[\frac{10}{2-\omega^2+j3\omega} + e^{-j\omega}\frac{10}{2-\omega^2+j3\omega}] = 10e^{-t}u(t) - 10e^{-2t}u(t) + 10e^{-(t-1)}u(t-1) - 10e^{-2(t-1)}u(t-$$
$$\Rightarrow \quad x(t) = \delta(t-2) + 10e^{-t}u(t) - 10e^{-2t}u(t) + 10e^{-(t-1)}u(t-1) - 10e^{-2(t-1)}u(t-1)$$

Problem 1b

$$x(t) = \delta(t-1) + \delta(t) + e^{-2(t+3)}u(t-1)$$
$$= \delta(t-1) + \delta(t) + e^{-8}\,e^{-2(t-1)}u(t-1)$$

TABLE AND DELAY PROPERTY GIVE

$$X(s) = e^{-s} + 1 + e^{-8}\,e^{-s}\cdot\frac{1}{s+2}$$
$$= 1 + e^{-s}\left[1 + \frac{e^{-8}}{s+2}\right]$$

Problem 2a

Note that the poles of $X(s)$ are 0 or negative, so $x(t)$ contains negative exponentials and a unit step, so the limit exists. Using L'Hospital's rule gives $\lim sX(s) = 1/3$.

Problem 3

For $x(t) = \cos(2t)u(t)$, generalized calculus gives

$$\dot{x}(t) = -2\sin(2t)u(t) + \cos(2t)\,\delta(t)$$
$$= -2\sin(2t)u(t) + \delta(t)$$

Then Tables give

$$\mathcal{L}[\dot{x}(t)] = -2\,\frac{2}{s^2+4} + 1 = \frac{s^2}{s^2+4}$$

On the other hand, the differentiation property gives

$$\mathcal{L}[\dot{x}(t)] = sX(s) - x(0^-) = s\,\frac{s}{s^2+4} + 0 = \frac{s^2}{s^2+4}$$