H10-08 Solutions

Prob 1

(a)

$$\begin{aligned}
x[n] &= 1+\cos(\frac{\pi}{3}n) \\
\Rightarrow \quad x[0] &= 2,\ x[1]=\frac{3}{2},\ x[2]=\frac{1}{2},\ x[3]=0,\ x[4]=\frac{1}{2},\ x[5]=-\frac{3}{2} \\
and \quad N_o &= 6,\ \omega_o=\frac{2\pi}{6} \\
x_k &= \frac{1}{N_o}\sum_{n=0}^{N_o-1} x[n]e^{-jk\omega_o n} = \frac{1}{6}\sum_{n=0}^{5} x[n]e^{-jk\frac{2\pi}{6}n} \\
&= \frac{1}{6}[2+\frac{3}{2}e^{-jk\frac{2\pi}{6}}+\frac{1}{2}e^{-jk\frac{4\pi}{6}}+0+\frac{1}{2}e^{-jk\frac{8\pi}{6}}+\frac{1}{2}e^{-jk\frac{10\pi}{6}}] \\
\Rightarrow \quad x_o &= 1,\ x_1=\frac{1}{2},\ x_2=x_3=x_4=0,\ x_5=\frac{1}{2}
\end{aligned}$$

And for other k values, $x_{k+6}=x_k$.

A shortcut is to write

$$x[n]=1+\frac{1}{2}e^{j\frac{\pi}{3}n}+\frac{1}{2}e^{-j\frac{\pi}{3}n}=1+\frac{1}{2}e^{j\frac{2\pi}{6}n}+\frac{1}{2}e^{-j\frac{2\pi}{6}n}$$

and compare to the DTFS expression to conclude

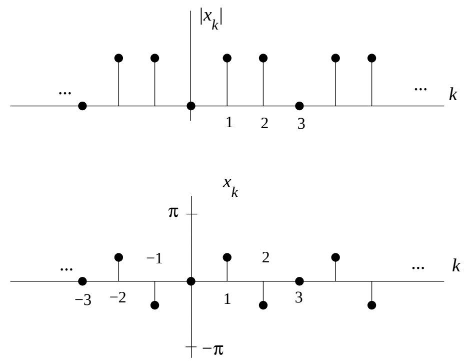
$$x_0=1,\ x_1=x_{-1}=\frac{1}{2},\quad x_2=x_3=x_4=0,\quad and\ \ x_{k+6}=x_k$$

(b)

$$\begin{aligned}
N_o &= 3,\ \omega_o=\frac{2\pi}{3} \\
x_k &= \frac{1}{N_o}\sum_{n=0}^{2} x[n]e^{-jk\omega_o n} = \frac{1}{3}[3-3e^{-jk\frac{2\pi}{3}}] \\
\Rightarrow \quad x_o &= 0,\ x_1=1-e^{-j\frac{2\pi}{3}},\ x_2=1-e^{-j\frac{4\pi}{3}}
\end{aligned}$$

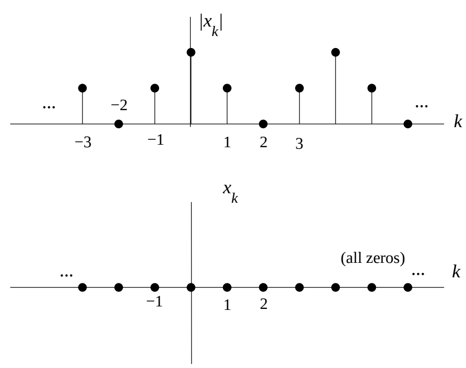
And for other k values, $x_{k+3}=x_k$.

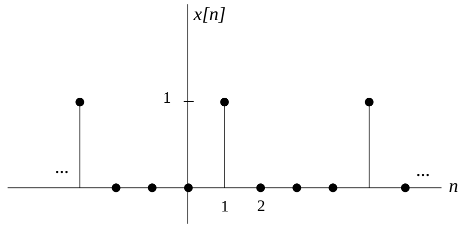
$$\begin{aligned}
x_1 &= 1-\cos\frac{2\pi}{3}+j\sin\frac{2\pi}{3}=\frac{3}{2}+j\frac{\sqrt{3}}{2} \quad \Rightarrow \quad |x_1|=\frac{\sqrt{6}}{2},\ \angle x_1\approx\frac{\pi}{5} \\
x_2 &= x_{-1}=x_1^*
\end{aligned}$$



(c)

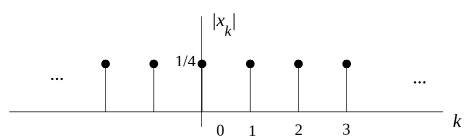
$$
\begin{aligned}
N_o &= 4 \quad \Rightarrow \quad \omega_o = \frac{\pi}{2} \\
x_k &= \frac{1}{N_o}\sum_{n=0}^{3} x[n] e^{-jk\frac{\pi}{2}n} \\
&= \frac{1}{4}[1 + \frac{\pi}{2}\underbrace{e^{-jk\frac{\pi}{2}}}_{(-j)^k} + 0 + \frac{\pi}{2}\underbrace{e^{-jk\frac{3\pi}{2}}}_{(j)^k}] \\
\Rightarrow \quad x_o &= \frac{1}{2},\ x_2 = 0,\ x_3 = \frac{1}{4},\ x_{k+4} = x_k
\end{aligned}
$$

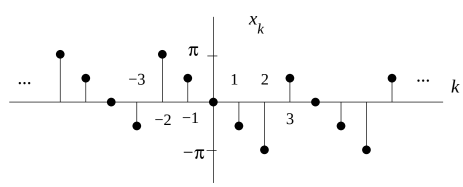




(d)

$$
\begin{aligned}
N_o &= 4 \quad \Rightarrow \quad \omega_o = \frac{\pi}{2} \\
x_k &= \frac{1}{4}\sum_{n=0}^{3} x[n] e^{-jk\frac{\pi}{2}n} = \frac{1}{4}[0 + e^{-jk\frac{\pi}{2}} + 0 + 0] \\
&= \frac{1}{4}(-j)^k, \ x_{k+4} = x_k \\
|x_k| &= \frac{1}{4}, \quad \angle x_k = -k\frac{\pi}{2} \\
\Rightarrow \quad \angle x_o &= 0, \ \angle x_1 = -\frac{\pi}{2}, \ \angle x_2 = \pi, \ \angle x_3 = -\frac{3\pi}{2} = \frac{\pi}{2}, \quad \textit{Sketch as odd function}
\end{aligned}
$$





Prob 2

(a)

$$
\begin{aligned}
\omega_o &= \pi \quad \Rightarrow \quad N_o = 2 \\
x_o &= \frac{1}{2}, \ x_1 = -\frac{1}{2} \\
x[n] &= \sum_{k=0}^{1} x_k e^{jk\omega_o n} = \frac{1}{2} - \frac{1}{2}e^{j\pi n} = \frac{1}{2} - \frac{1}{2}(-1)^n
\end{aligned}
$$

$$= \begin{cases} 0, & n\ even \\ 1, & n\ odd \end{cases}$$

(b)

$$\begin{aligned} x[n] &= \sum_{k=0}^{1} x_k e^{jk\omega_o n} = \sum_{k=0}^{1} \frac{1}{2} e^{jk\pi n} = \frac{1}{2} + \frac{1}{2}(-1)^n \\ &= \begin{cases} 1, & n\ even \\ 0, & n\ odd \end{cases} \end{aligned}$$

(c)

$$\begin{aligned} \omega_o &= \frac{\pi}{3} \quad \Rightarrow \quad N_o = 6 \\ x[n] &= \sum_{k=0}^{5} x_k e^{jk\frac{\pi}{3}n} \\ &= -1 + e^{j\frac{2\pi}{3}n} - 2e^{j\pi n} + e^{j\frac{4\pi}{3}n} \\ &= -1 + e^{j\frac{2\pi}{3}n} - 2(-1)^n + e^{-j\frac{2\pi}{3}n} \\ &= -1 - 2(-1)^n + 2\cos(\frac{2\pi}{3}n) \end{aligned}$$

Prob 3

$$\begin{aligned} x[n] &= -x[n - \frac{N_o}{2}] \quad , N_o\ even \\ x_k &= \frac{1}{N_o} \sum_{n=0}^{N_o-1} x[n] e^{-jk\omega_o n} \\ &= \frac{1}{N_o} \sum_{n=0}^{\frac{N_o}{2}-1} x[n] e^{-jk\omega_o n} + \frac{1}{N_o} \sum_{n=\frac{N_o}{2}}^{N_o-1} x[n] e^{-jk\omega_o n} \end{aligned}$$

Let $m = n - \frac{N_o}{2}$

$$x_k = \frac{1}{N_o} \sum_{n=0}^{\frac{N_o}{2}-1} x[n] e^{-jk\omega_o n} + \frac{1}{N_o} \sum_{m=0}^{\frac{N_o}{2}-1} x[m + \frac{N_o}{2}] e^{-jk\omega_o (m + \frac{N_o}{2})}$$

But $x[m + \frac{N_o}{2}] = -x[m]$,
and $e^{-jk\omega_o \frac{N_o}{2}} = e^{-jk\pi} = (-1)^k$

$$\begin{aligned} \Rightarrow \quad x_k &= \frac{1}{N_o} \sum_{n=0}^{\frac{N_o}{2}-1} [x[n] - (-1)^k x[n]] e^{-jk\omega_o n} \\ &= 0, if\ k\ is\ even. \end{aligned}$$

Prob 4

$$
\begin{aligned}
x[n] &= \sum_{k-0}^{4} x_k e^{jk\frac{2\pi}{5}n} \\
&= -2e^{j\frac{2\pi}{5}n} - 2e^{j\frac{8\pi}{5}n} \\
&= -2e^{j\frac{2\pi}{5}n} - 2e^{-j\frac{2\pi}{5}n} \\
&= -4\cos(\frac{2\pi}{5}n)
\end{aligned}
$$

Prob 5

(a)

$$
\begin{aligned}
\hat{N}_o &= N_o,\ \hat{\omega}_o \\
\hat{x}_k &= \frac{1}{N_o}\sum_{n=0}^{N_o-1} \hat{x}[n]e^{-jk\omega_o n} \\
&= \frac{1}{N_o}\sum_{n=0}^{N_o-1} x[n+\frac{N_o}{2}]e^{-jk\omega_o n}
\end{aligned}
$$

Let $m = n + \frac{N_o}{2}$

$$
\begin{aligned}
\hat{x}_k &= \frac{1}{N_o}\sum_{m=N_o/2}^{N_o/2+N_o-1} x[m]e^{-jk\omega_o(m-N_o/2)} \\
&= e^{-jk\omega_o\frac{N_o}{2}}\frac{N_o}{2}\sum_{m=<N_o>} x[m]e^{-jk\omega_o m}
\end{aligned}
$$

But $\omega_o\frac{N_o}{2} = \pi$, so

$$\hat{x}_k = (-1)^k x_k$$

(b) Assume $\hat{N}_o = N_o$, $\hat{\omega}_o = \omega_o$. (This might not be the case, e.g. $x[n] = (-1)^n$). Then

$$
\begin{aligned}
\hat{x}_k &= \frac{1}{N_o}\sum_{n=0}^{N_o-1} (-1)^n x[n]e^{-jk\omega_o n} \\
&= \frac{1}{N_o}\sum_{n=0}^{N_o-1} x[n]e^{j\pi n}e^{-jk\omega_o n} \\
&= \frac{1}{N_o}\sum_{n=0}^{N_o-1} x[n]e^{-j\frac{\omega_o N_o}{2}n}e^{jk\omega_o n} \\
&= \frac{1}{N_o}\sum_{n=0}^{N_o-1} x[n]e^{-j(k-\frac{N_o}{2})\omega_o n} \\
&= x_{k-\frac{N_o}{2}}
\end{aligned}
$$